19UEC904 – CONTROL ENGINEERING

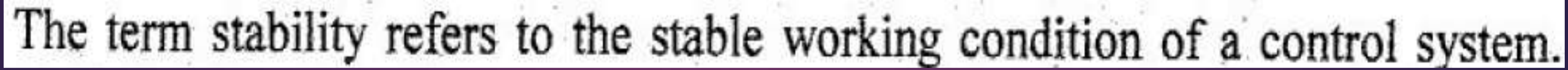
UNIT IV - STABILITY ANALYSIS

Concept of stability, Bounded - Input Bounded - Output stability, Routh stability criterion, Root locus concept-Guidelines for sketching root locus, Nyquist stability criterion.

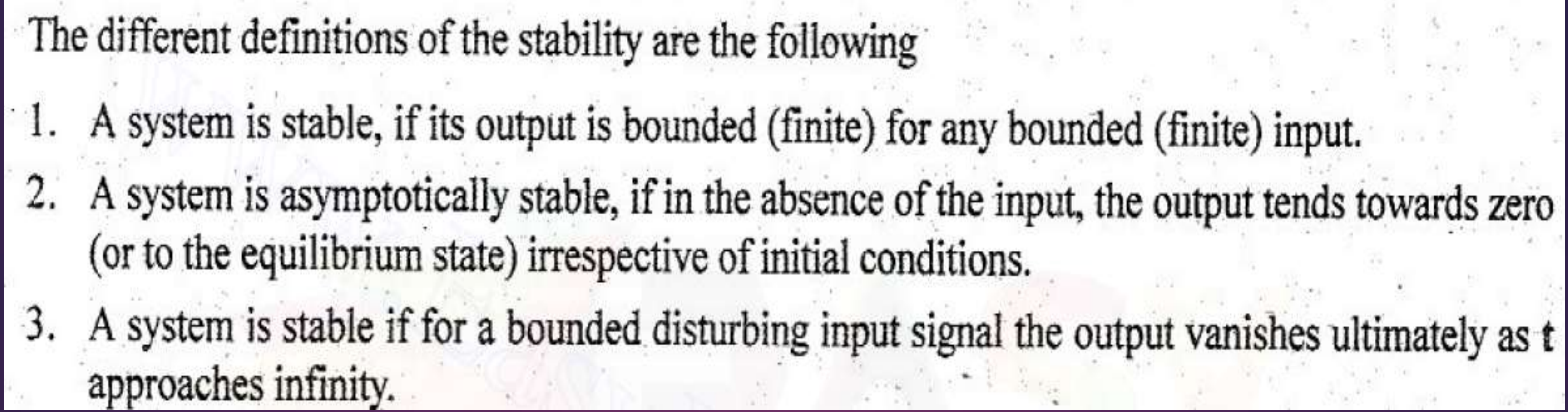
Topics to be covered

- Stability, Routh-Hurwitz Criterion
- Root Locus Technique
- Nyquist Stability Criterion
- Relative Stability

STABILITY

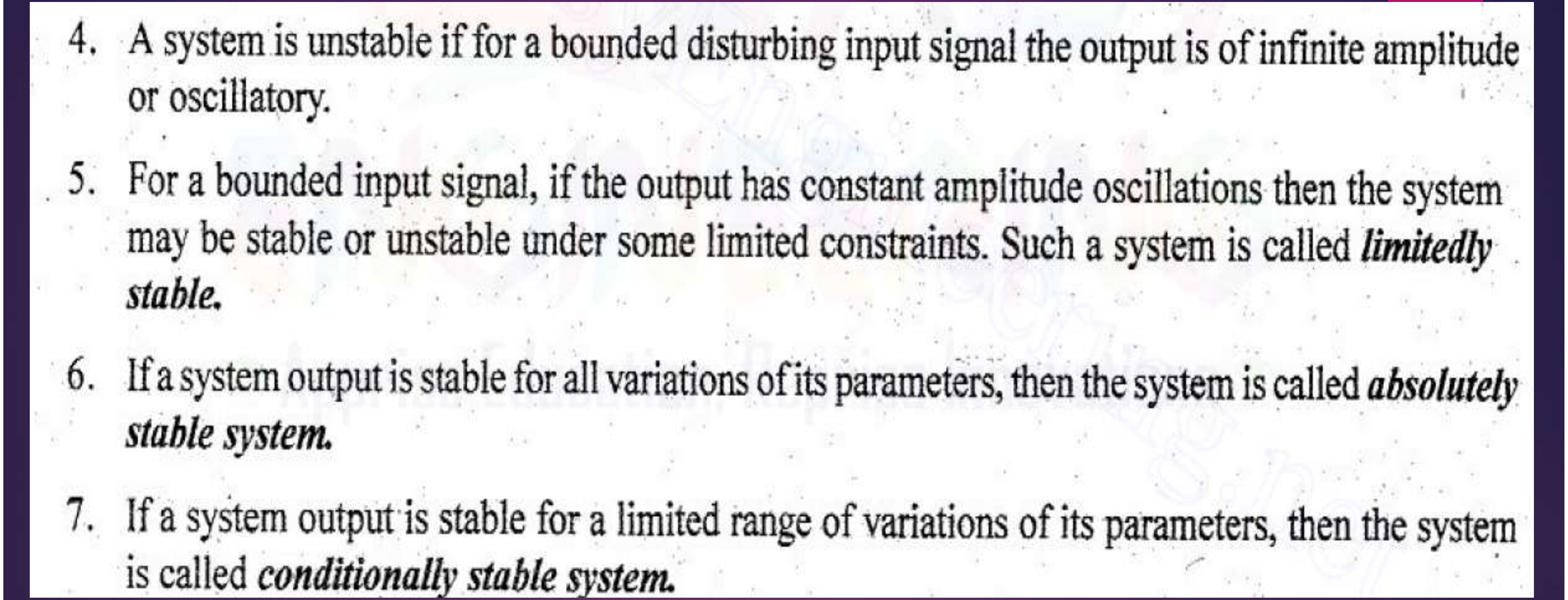


The term stability refers to the stable working condition of a control system.



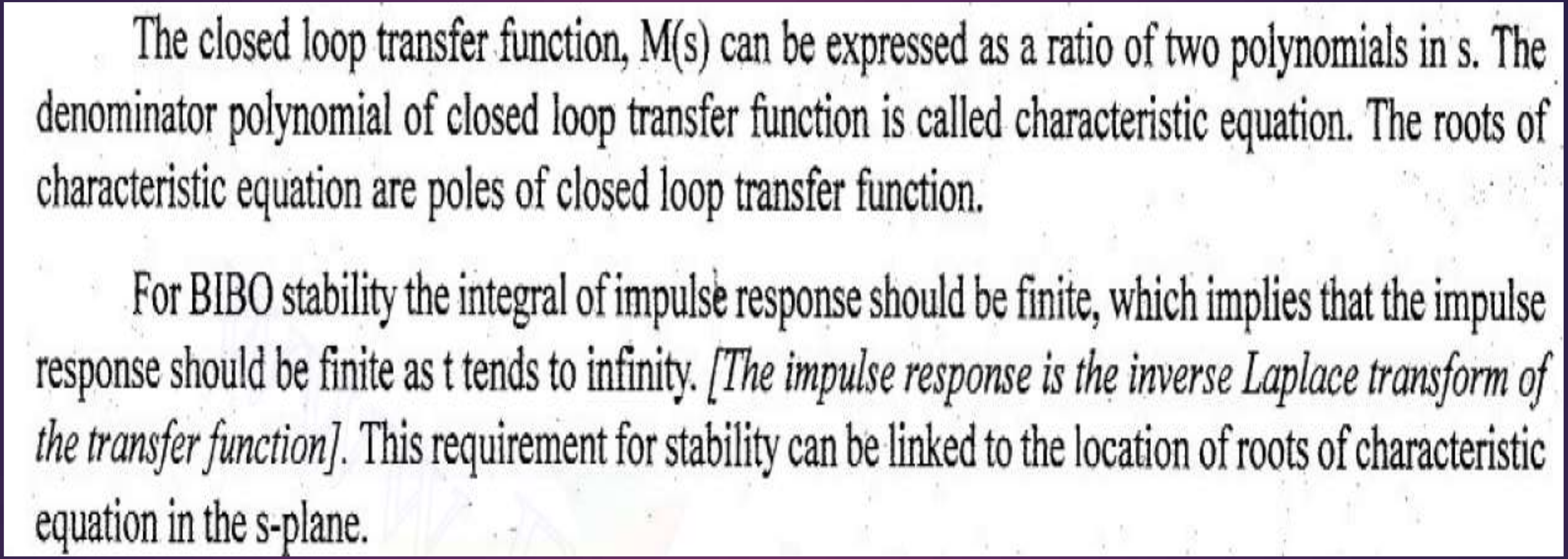
The different definitions of the stability are the following

1. A system is stable, if its output is bounded (finite) for any bounded (finite) input.
2. A system is asymptotically stable, if in the absence of the input, the output tends towards zero (or to the equilibrium state) irrespective of initial conditions.
3. A system is stable if for a bounded disturbing input signal the output vanishes ultimately as t approaches infinity.



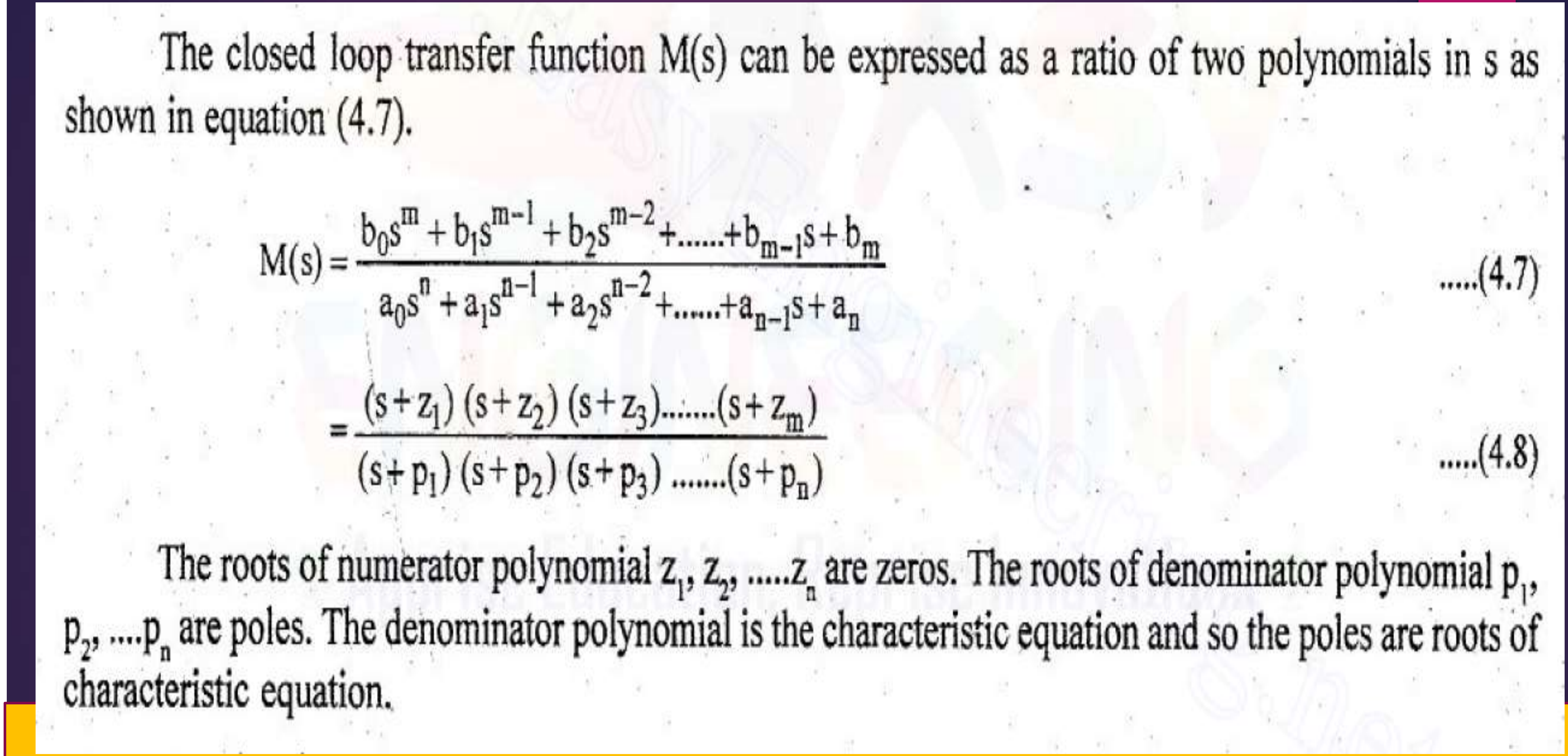
4. A system is unstable if for a bounded disturbing input signal the output is of infinite amplitude or oscillatory.
5. For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called ***limitedly stable.***
6. If a system output is stable for all variations of its parameters, then the system is called ***absolutely stable system.***
7. If a system output is stable for a limited range of variations of its parameters, then the system is called ***conditionally stable system.***

LOCATION OF POLES ON s-PLANE FOR STABILITY



The closed loop transfer function, M(s) can be expressed as a ratio of two polynomials in s. The denominator polynomial of closed loop transfer function is called characteristic equation. The roots of characteristic equation are poles of closed loop transfer function.

For BIBO stability the integral of impulse response should be finite, which implies that the impulse response should be finite as t tends to infinity. *[The impulse response is the inverse Laplace transform of the transfer function]*. This requirement for stability can be linked to the location of roots of characteristic equation in the s-plane.

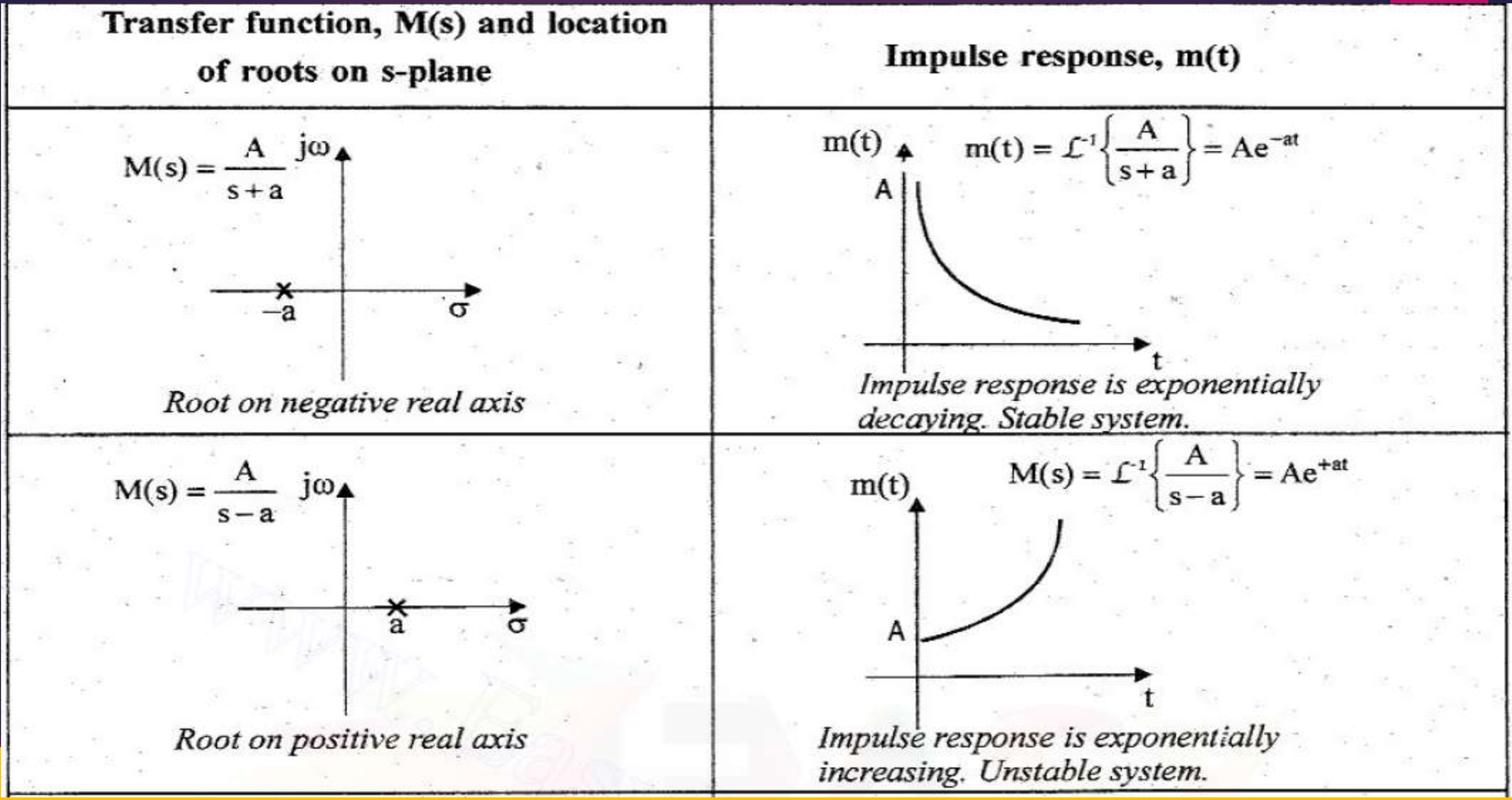


The closed loop transfer function M(s) can be expressed as a ratio of two polynomials in s as shown in equation (4.7).

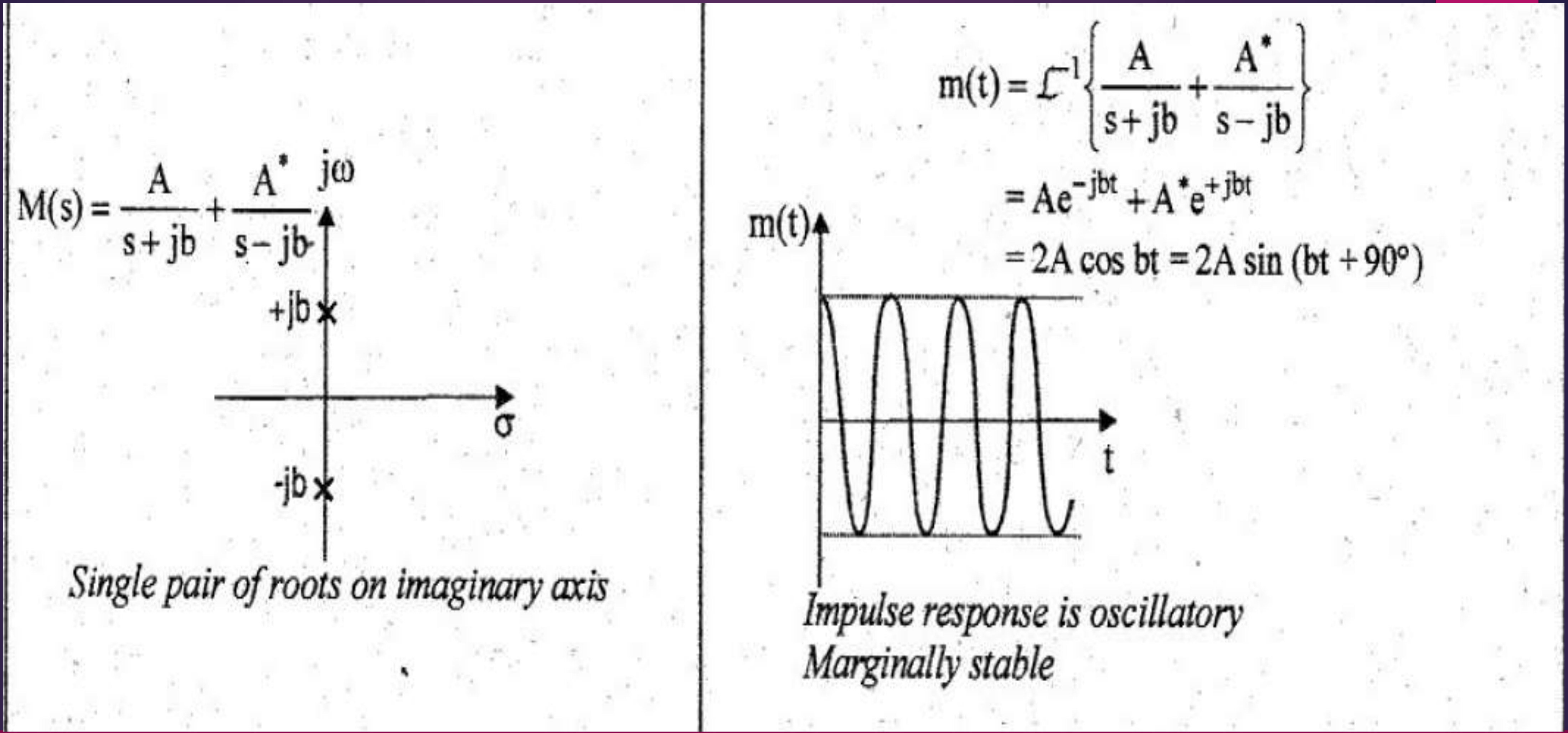
$$M(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + ...+ b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...+ a_{n-1} s + a_n} \qquad .....(4.7)$$

$$= \frac{(s+z_1)\,(s+z_2)\,(s+z_3)........(s+z_m)}{(s+p_1)\,(s+p_2)\,(s+p_3)\,........(s+p_n)} \qquad .....(4.8)$$

The roots of numerator polynomial z_1, z_2, ...z_n are zeros. The roots of denominator polynomial p_1, p_2, ...p_n are poles. The denominator polynomial is the characteristic equation and so the poles are roots of characteristic equation.



Transfer function, M(s) and location of roots on s-plane	Impulse response, m(t)
$M(s) = \frac{A}{s+a}$ *Root on negative real axis*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+a}\right\} = Ae^{-at}$ *Impulse response is exponentially decaying. Stable system.*
$M(s) = \frac{A}{s-a}$ *Root on positive real axis*	$M(s) = \mathcal{L}^{-1}\left\{\frac{A}{s-a}\right\} = Ae^{+at}$ *Impulse response is exponentially increasing. Unstable system.*



$$M(s) = \frac{A}{s+jb} + \frac{A^*}{s-jb}$$

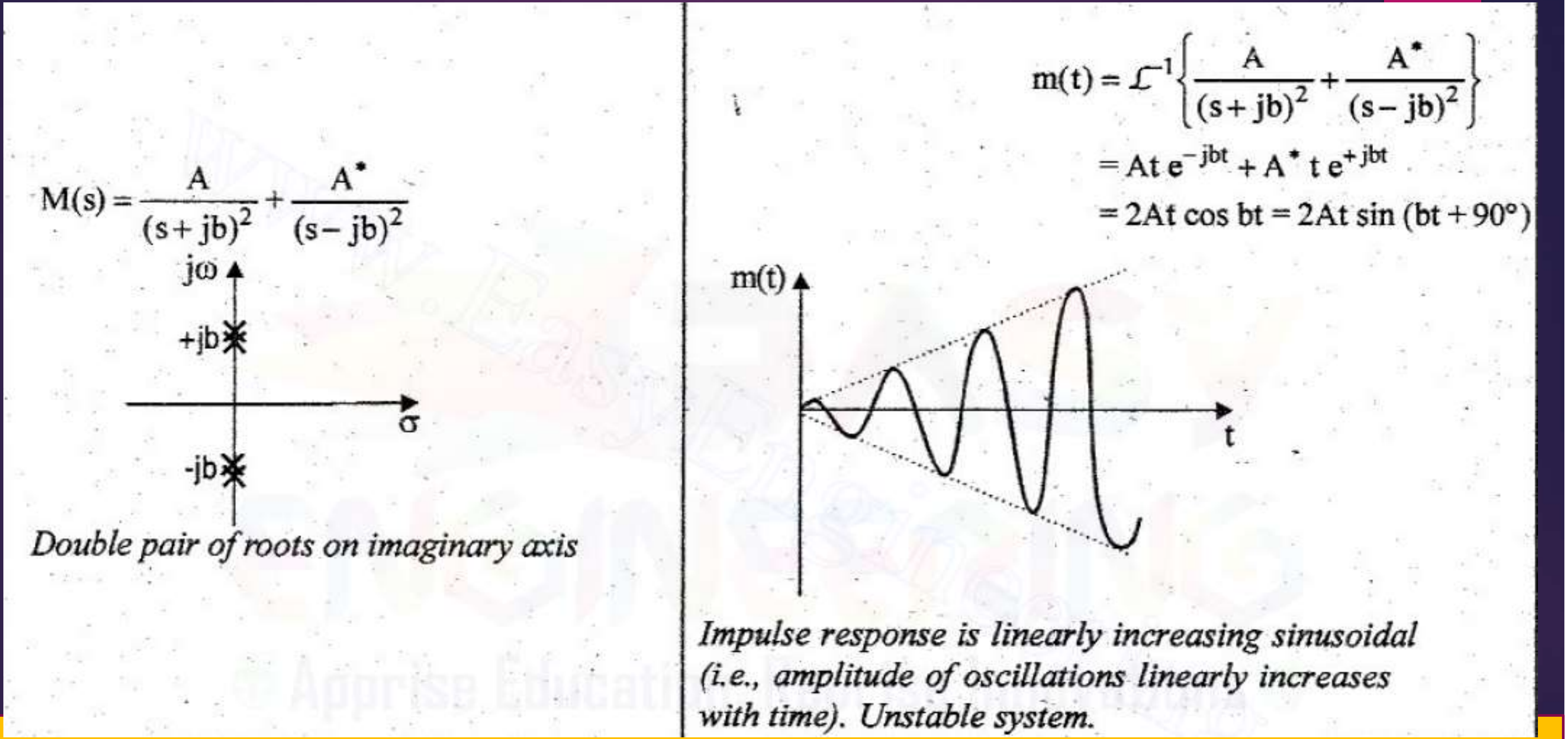
Single pair of roots on imaginary axis

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+jb} + \frac{A^*}{s-jb}\right\}$$

$$= Ae^{-jbt} + A^*e^{+jbt}$$

$$= 2A\cos bt = 2A\sin(bt + 90^\circ)$$

Impulse response is oscillatory
Marginally stable



$$M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$$

Double pair of roots on imaginary axis

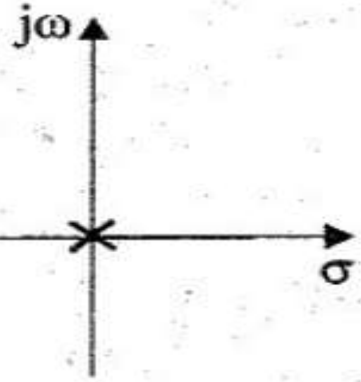
$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}\right\}$$

$$= At\,e^{-jbt} + A^*\,t\,e^{+jbt}$$

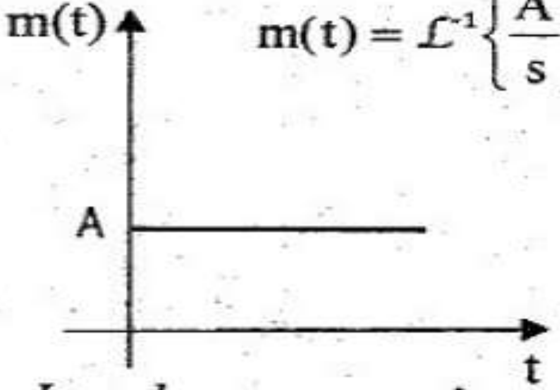
$$= 2At\cos bt = 2At\sin(bt + 90°)$$

Impulse response is linearly increasing sinusoidal (i.e., amplitude of oscillations linearly increases with time). Unstable system.

$$M(s) = \frac{A}{s}$$



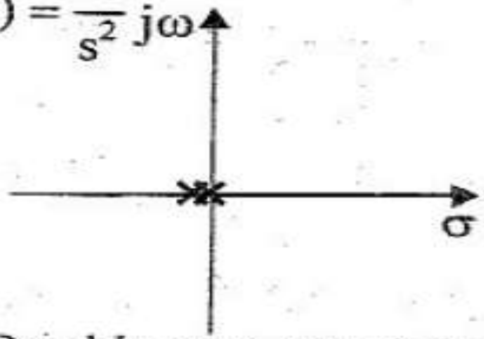
Single root at origin



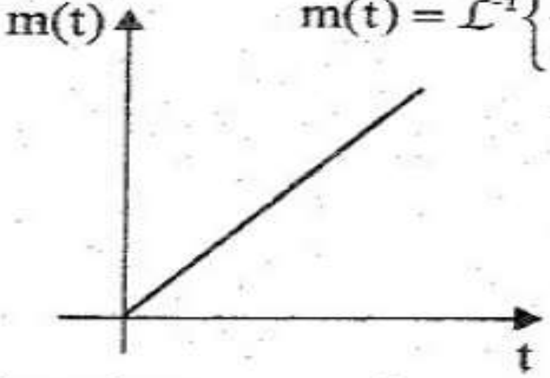
$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s}\right\} = A$$

Impulse response is constant.
Marginally stable system.

$$M(s) = \frac{A}{s^2}$$

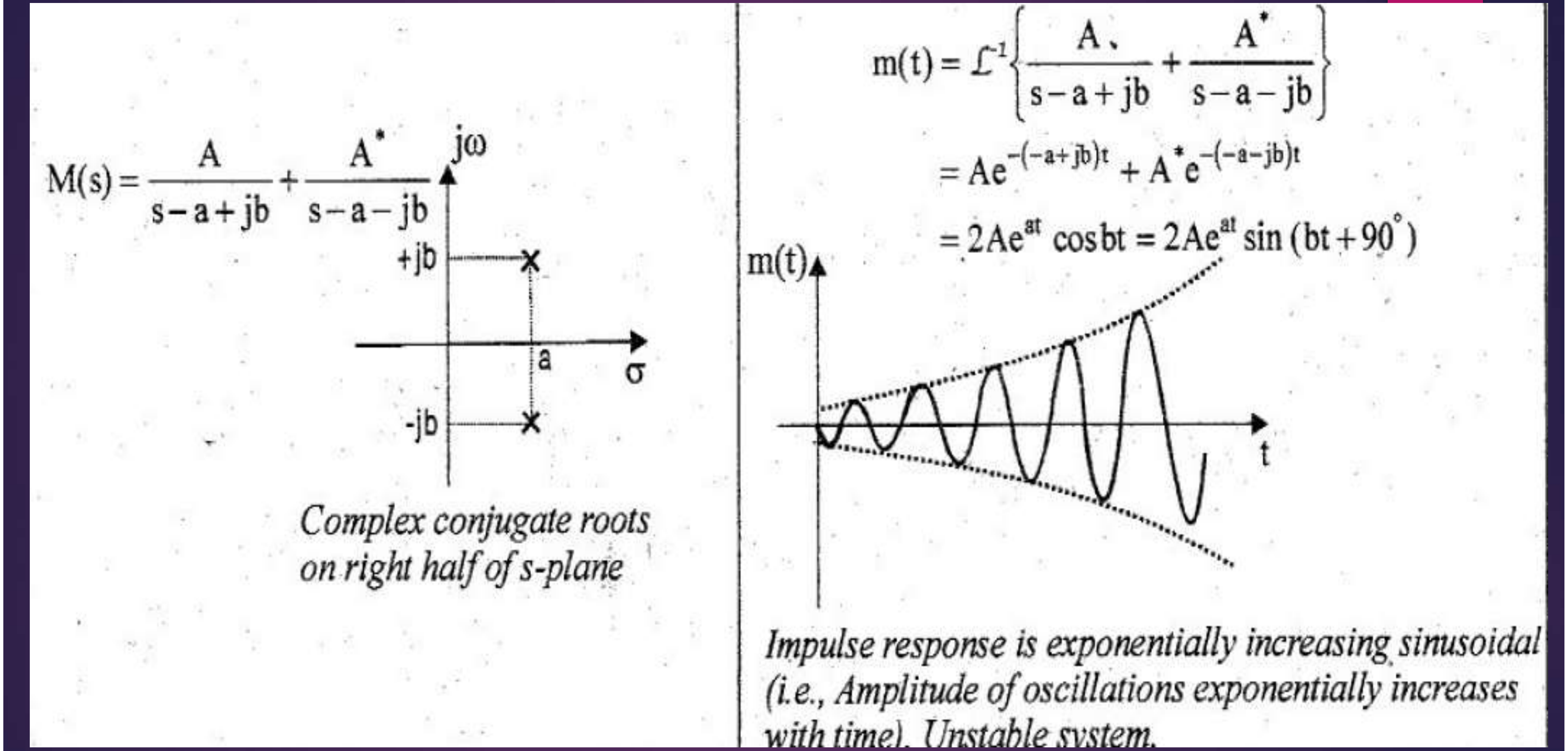


Double root at origin



$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s^2}\right\} = At$$

Impulse reponse linearly increases with time. Unstable system



$$M(s) = \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}$$

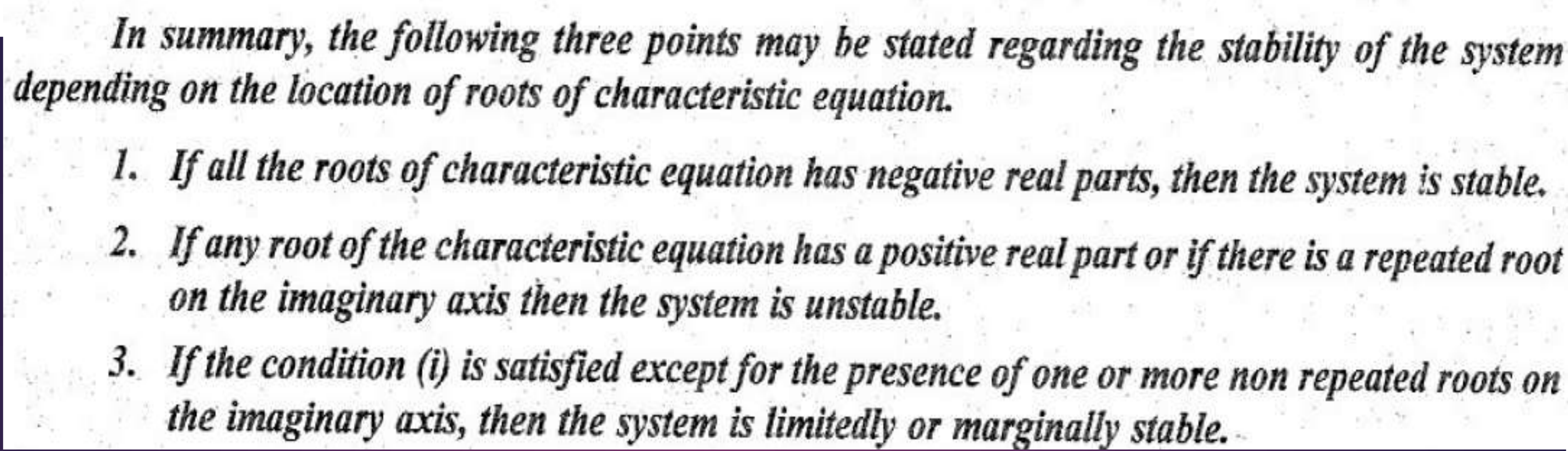
Complex conjugate roots on right half of s-plane

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}\right\}$$

$$= Ae^{-(-a+jb)t} + A^*e^{-(-a-jb)t}$$

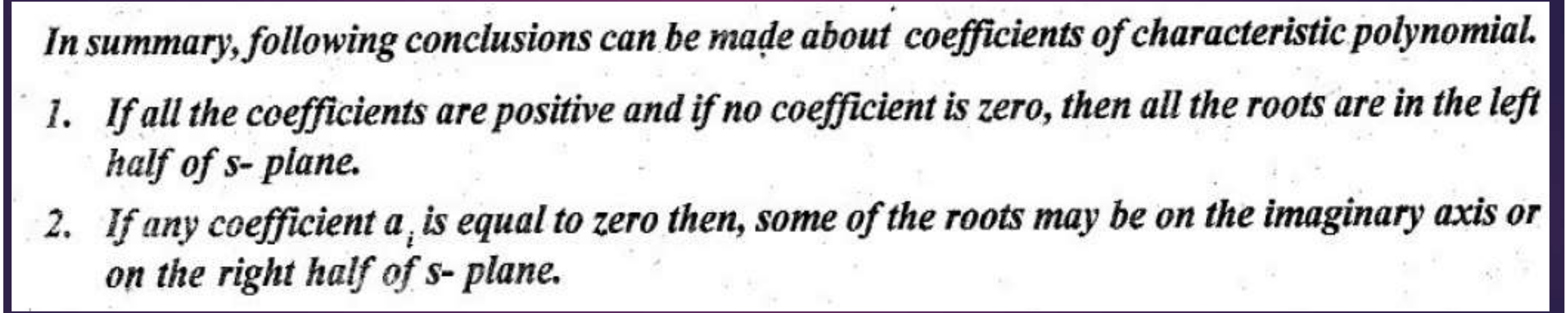
$$= 2Ae^{at}\cos bt = 2Ae^{at}\sin(bt+90^\circ)$$

Impulse response is exponentially increasing sinusoidal (i.e., Amplitude of oscillations exponentially increases with time). Unstable system.



In summary, the following three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation.

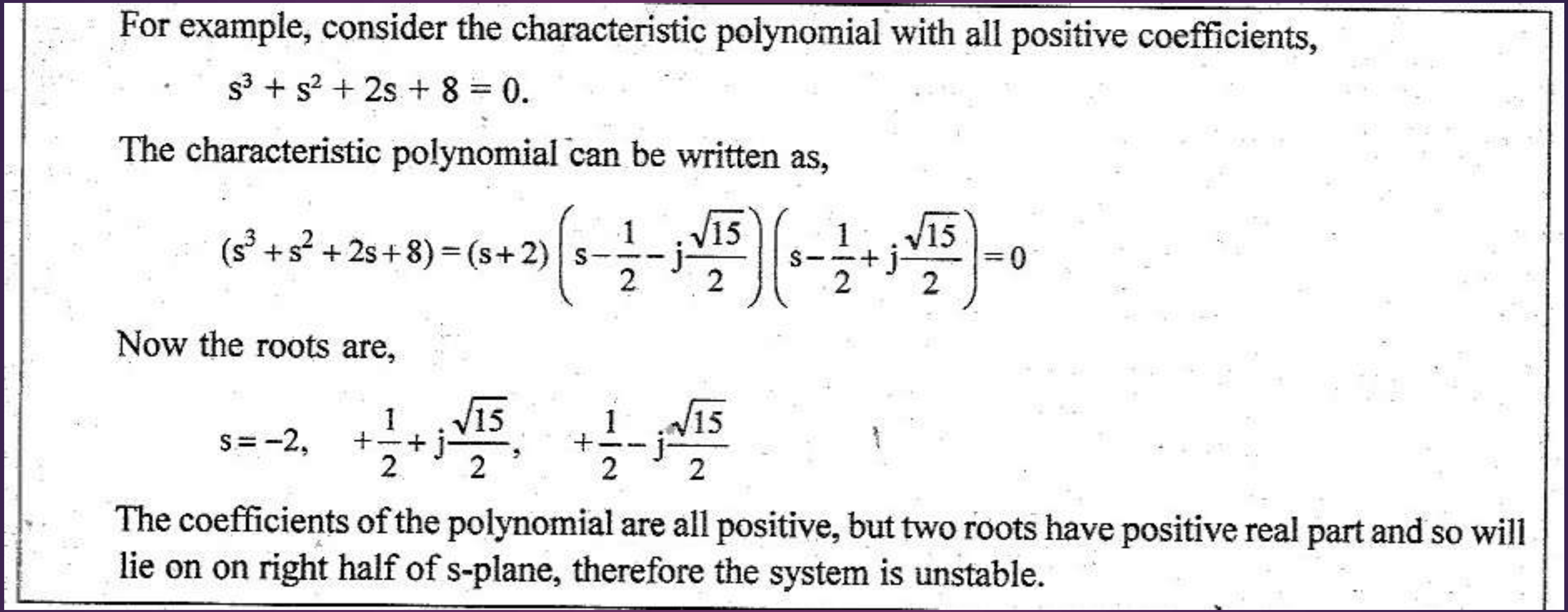
1. *If all the roots of characteristic equation has negative real parts, then the system is stable.*
2. *If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis then the system is unstable.*
3. *If the condition (i) is satisfied except for the presence of one or more non repeated roots on the imaginary axis, then the system is limitedly or marginally stable.*



In summary, following conclusions can be made about coefficients of characteristic polynomial.

1. ***If all the coefficients are positive and if no coefficient is zero, then all the roots are in the left half of s- plane.***
2. ***If any coefficient a_i is equal to zero then, some of the roots may be on the imaginary axis or on the right half of s- plane.***

3. ***If any coefficient a_i is negative then atleast one root is in the right half of s- plane.***



For example, consider the characteristic polynomial with all positive coefficients,

$$s^3 + s^2 + 2s + 8 = 0.$$

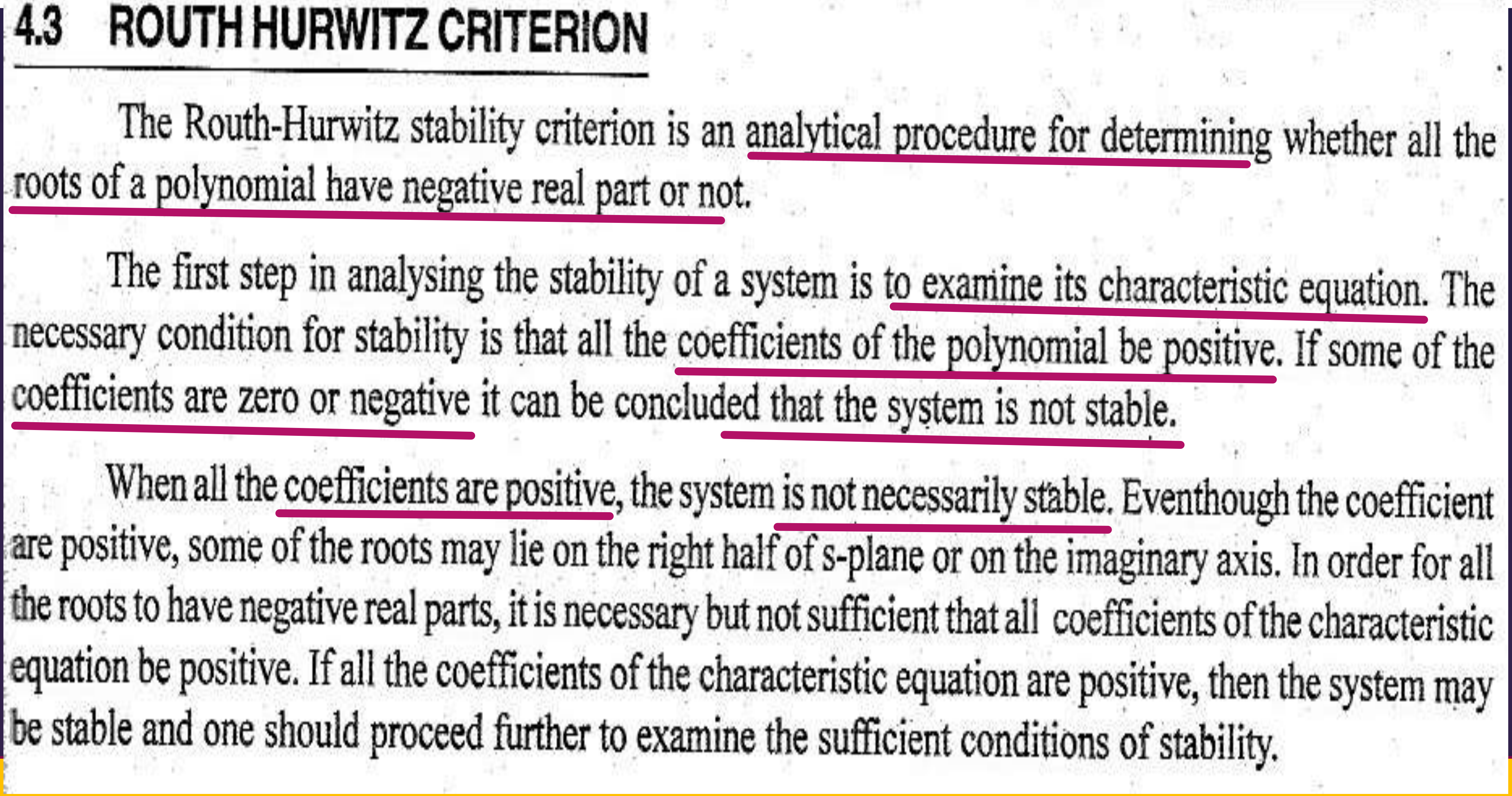
The characteristic polynomial can be written as,

$$(s^3+s^2+2s+8)=(s+2)\left(s-\frac{1}{2}-j\frac{\sqrt{15}}{2}\right)\left(s-\frac{1}{2}+j\frac{\sqrt{15}}{2}\right)=0$$

Now the roots are,

$$s=-2,\quad +\frac{1}{2}+j\frac{\sqrt{15}}{2},\quad +\frac{1}{2}-j\frac{\sqrt{15}}{2}$$

The coefficients of the polynomial are all positive, but two roots have positive real part and so will lie on on right half of s-plane, therefore the system is unstable.

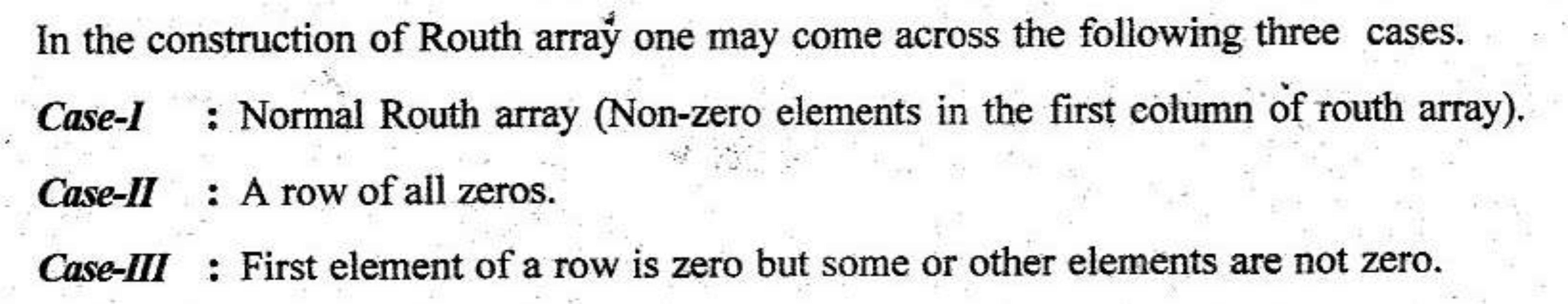


4.3 ROUTH HURWITZ CRITERION

The Routh-Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial have negative real part or not.

The first step in analysing the stability of a system is to examine its characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial be positive. If some of the coefficients are zero or negative it can be concluded that the system is not stable.

When all the coefficients are positive, the system is not necessarily stable. Eventhough the coefficient are positive, some of the roots may lie on the right half of s-plane or on the imaginary axis. In order for all the roots to have negative real parts, it is necessary but not sufficient that all coefficients of the characteristic equation be positive. If all the coefficients of the characteristic equation are positive, then the system may be stable and one should proceed further to examine the sufficient conditions of stability.



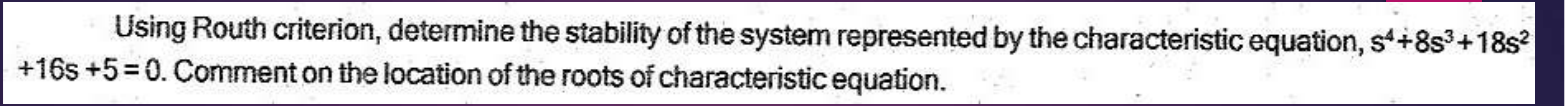
In the construction of Routh array one may come across the following three cases.

Case-I : Normal Routh array (Non-zero elements in the first column of routh array).

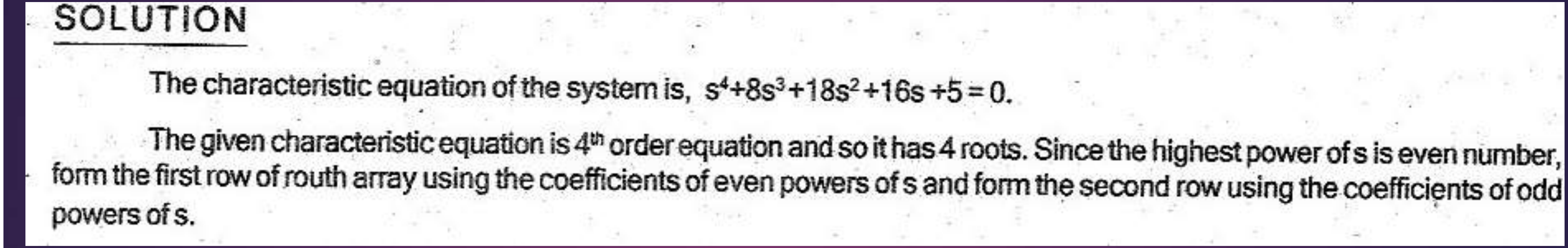
Case-II : A row of all zeros.

Case-III : First element of a row is zero but some or other elements are not zero.

EX NO:#1



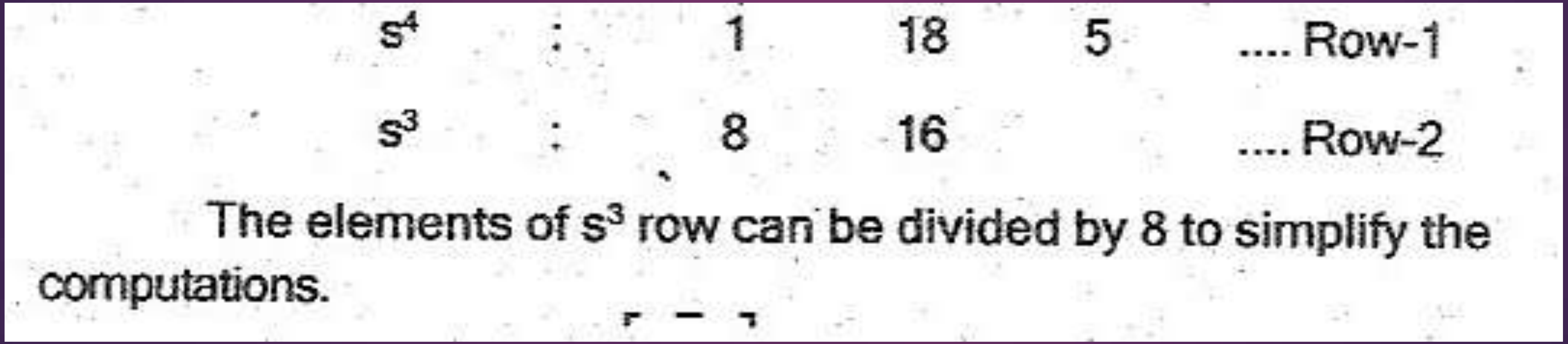
Using Routh criterion, determine the stability of the system represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.



SOLUTION

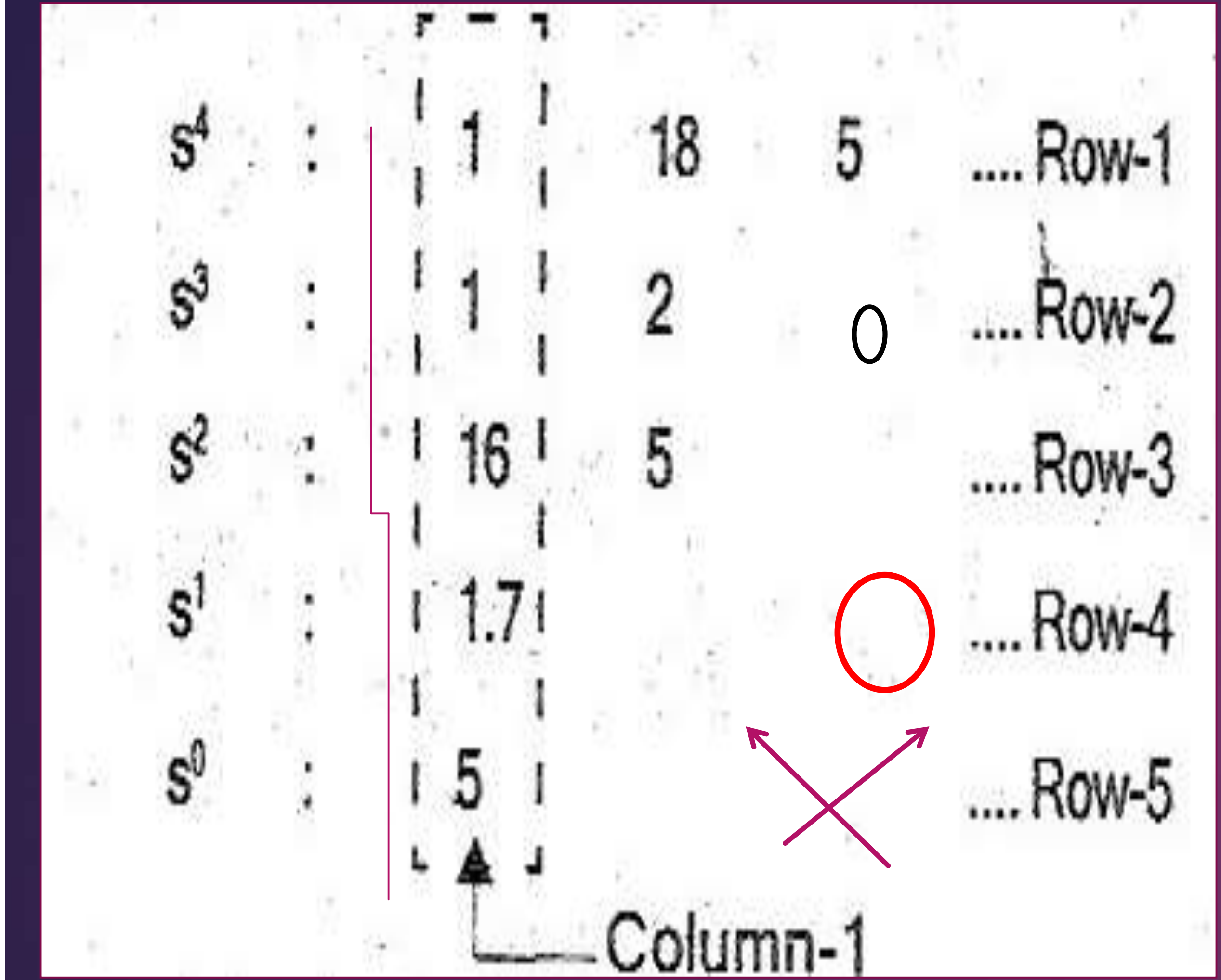
The characteristic equation of the system is, $s^4+8s^3+18s^2+16s+5=0$.

The given characteristic equation is 4th order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

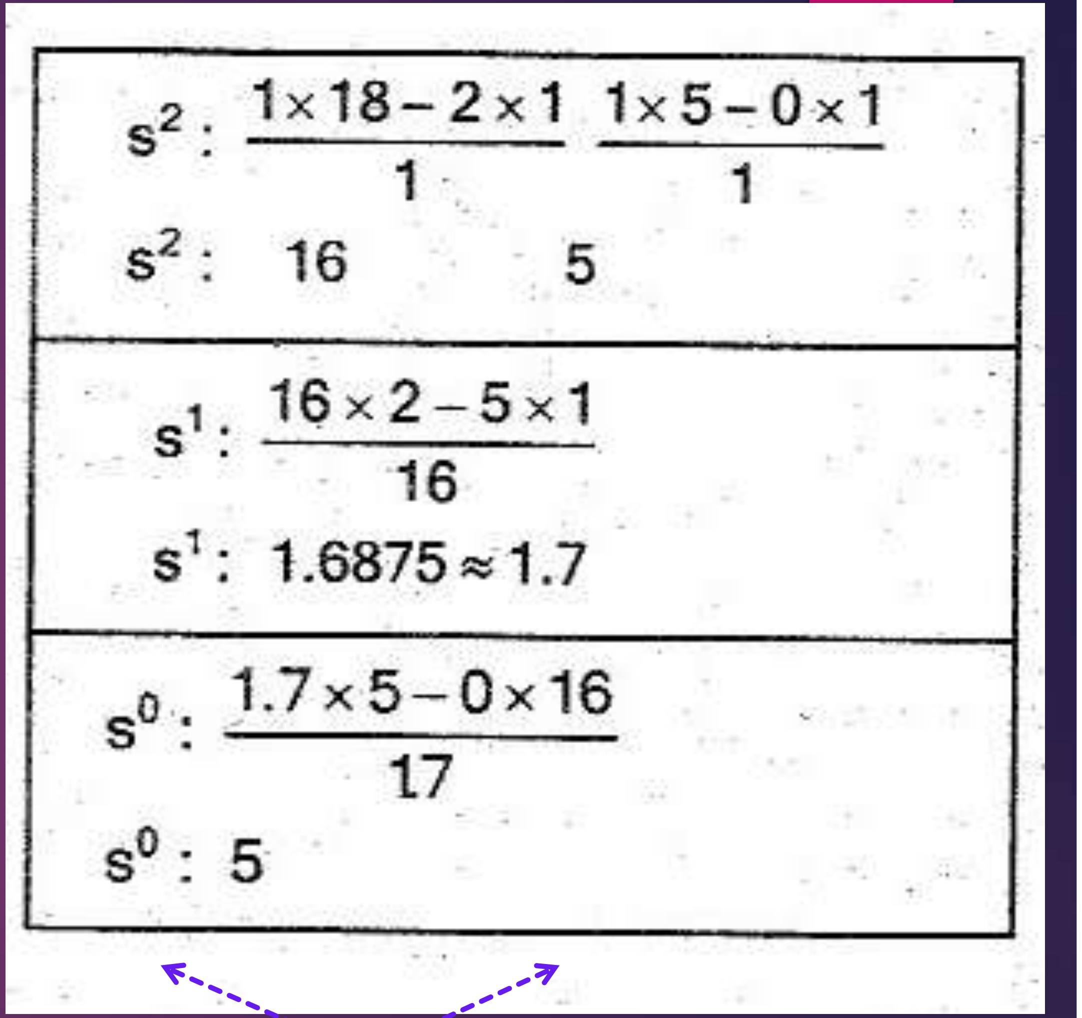


s^4	:	1	18	5	.... Row-1
s^3	:	8	16		.... Row-2

The elements of s^3 row can be divided by 8 to simplify the computations.



s^4	:	1	18	5	.... Row-1
s^3	:	1	2	0	.... Row-2
s^2	:	16	5		.... Row-3
s^1	:	1.7			.... Row-4
s^0	:	5			.... Row-5



$$s^2 : \frac{1\times18-2\times1}{1} \quad \frac{1\times5-0\times1}{1}$$

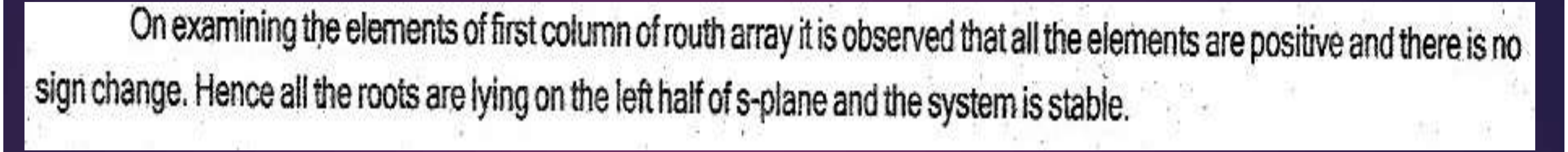
$$s^2 : \quad 16 \qquad 5$$

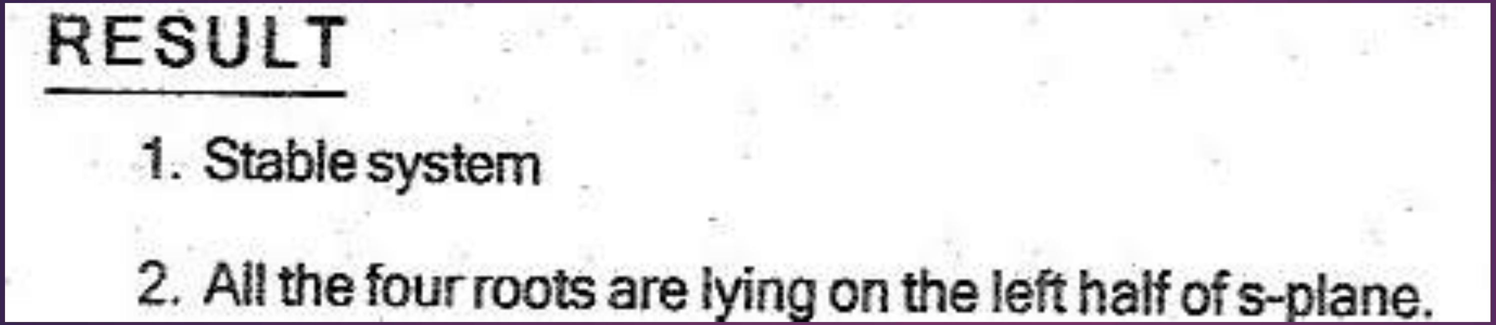
$$s^1 : \frac{16\times2-5\times1}{16}$$

$$s^1 : \ 1.6875\approx1.7$$

$$s^0 : \frac{1.7\times5-0\times16}{17}$$

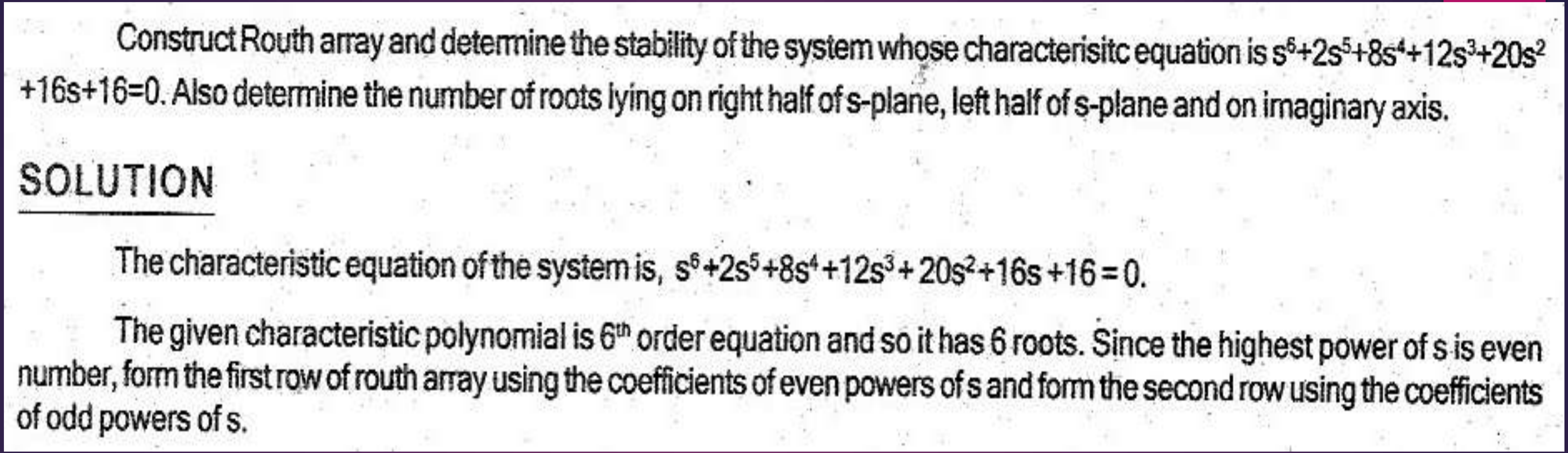
$$s^0 : \ 5$$


On examining the elements of first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.


RESULT

1. Stable system
2. All the four roots are lying on the left half of s-plane.

EX NO:#2

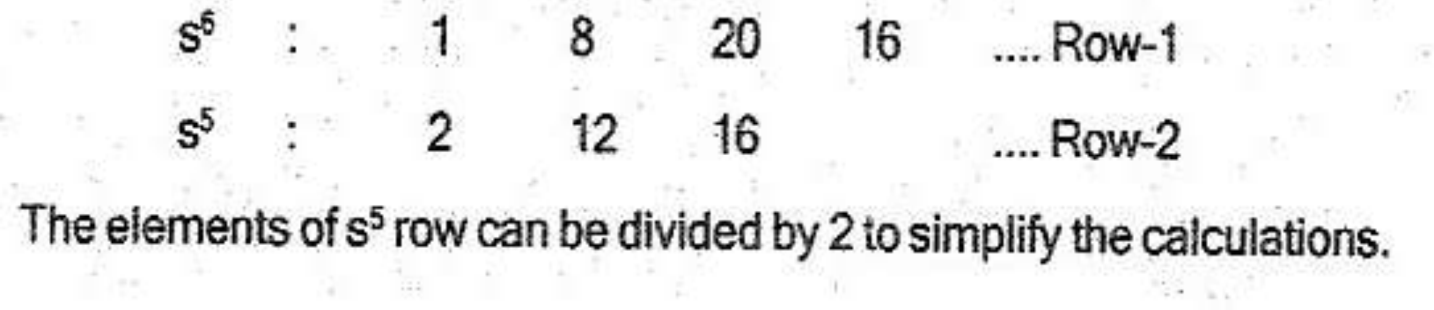


Construct Routh array and determine the stability of the system whose characterisitc equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

SOLUTION

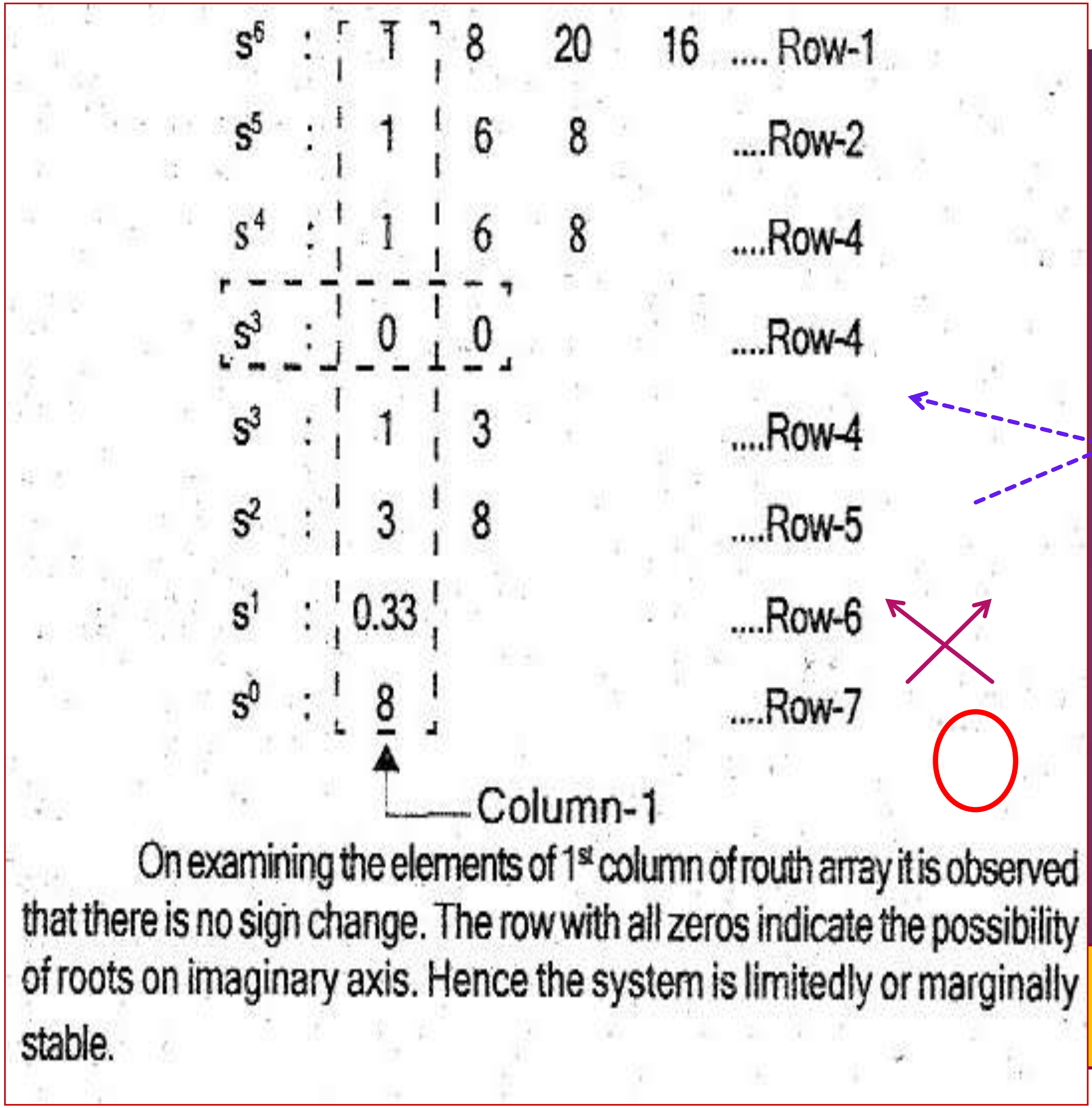
The characteristic equation of the system is, $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.



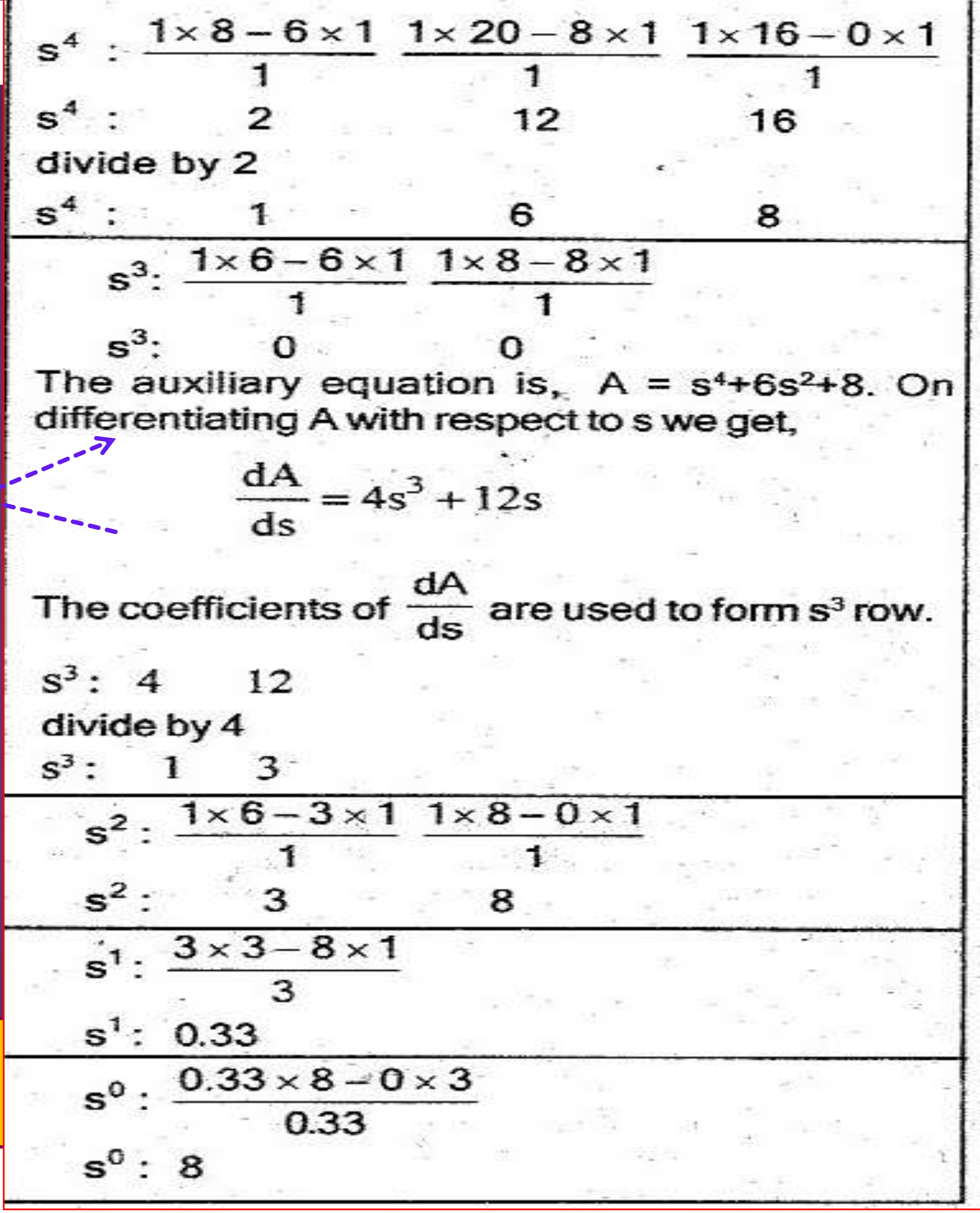
s^6	:	1	8	20	16	.... Row-1
s^5	:	2	12	16		.... Row-2

The elements of s^5 row can be divided by 2 to simplify the calculations.



s^6	:	1	8	20	16	....Row-1
s^5	:	1	6	8		....Row-2
s^4	:	1	6	8		....Row-4
s^3	:	0	0			....Row-4
s^3	:	1	3			....Row-4
s^2	:	3	8			....Row-5
s^1	:	0.33				....Row-6
s^0	:	8				....Row-7

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.



s^4 : $\frac{1\times8-6\times1}{1}$ $\frac{1\times20-8\times1}{1}$ $\frac{1\times16-0\times1}{1}$

s^4 : 2 12 16

divide by 2

s^4 : 1 6 8

s^3: $\frac{1\times6-6\times1}{1}$ $\frac{1\times8-8\times1}{1}$

s^3: 0 0

The auxiliary equation is, $A = s^4+6s^2+8$. On differentiating A with respect to s we get,

$$\frac{dA}{ds} = 4s^3 + 12s$$

The coefficients of $\frac{dA}{ds}$ are used to form s^3 row.

s^3: 4 12

divide by 4

s^3: 1 3

s^2 : $\frac{1\times6-3\times1}{1}$ $\frac{1\times8-0\times1}{1}$

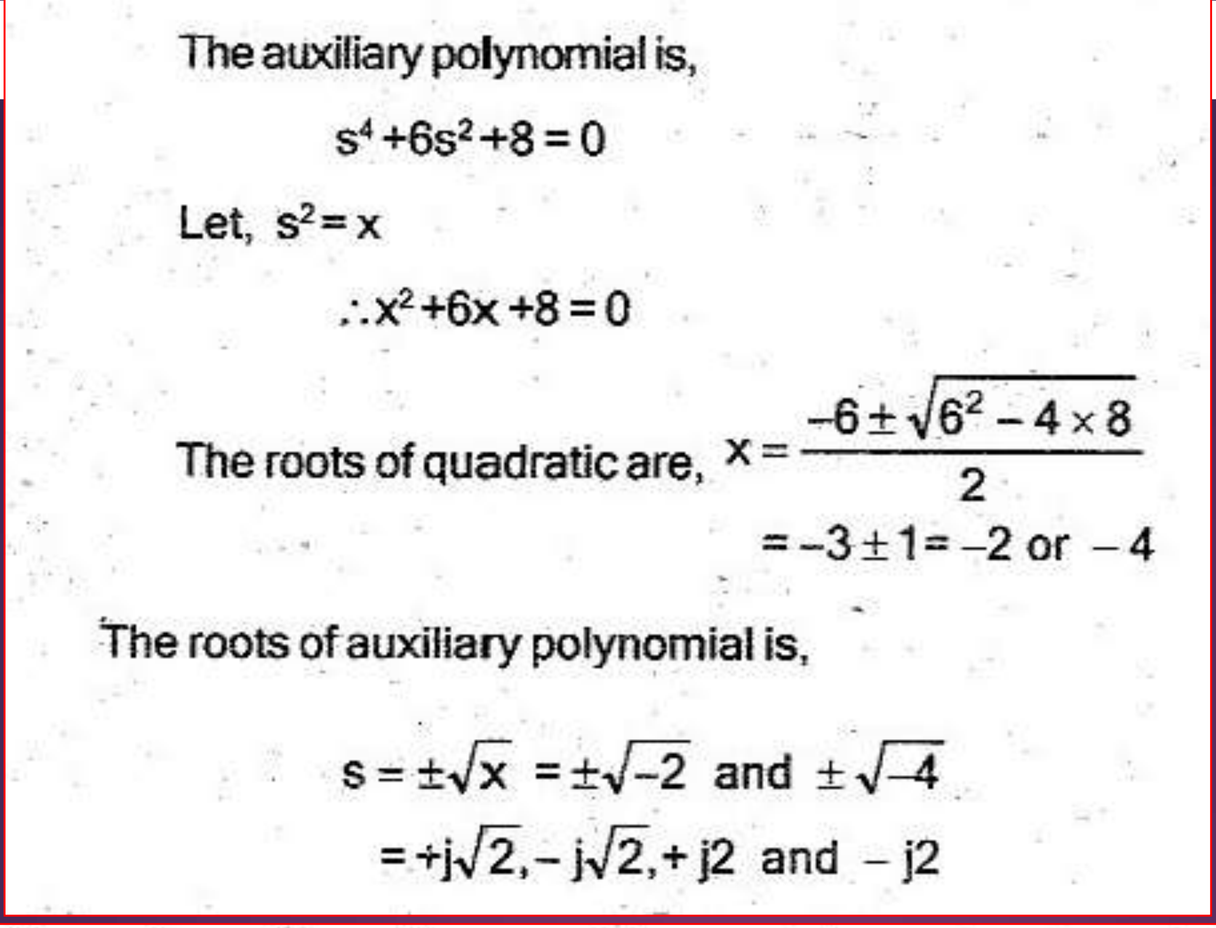
s^2 : 3 8

s^1 : $\frac{3\times3-8\times1}{3}$

s^1 : 0.33

s^0 : $\frac{0.33\times8-0\times3}{0.33}$

s^0 : 8



The auxiliary polynomial is,

$$s^4+6s^2+8=0$$

Let, $s^2=x$

$$\therefore x^2+6x+8=0$$

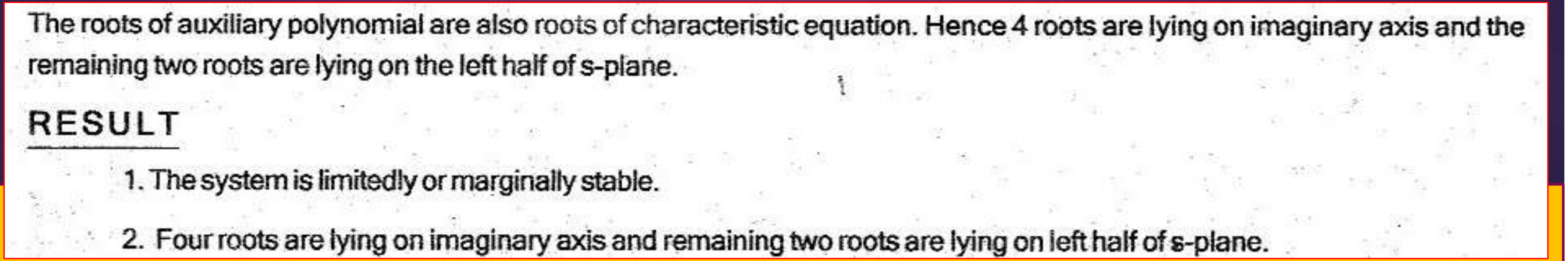
The roots of quadratic are, $x=\dfrac{-6\pm\sqrt{6^2-4\times 8}}{2}$

$$=-3\pm 1=-2 \text{ or } -4$$

The roots of auxiliary polynomial is,

$$s=\pm\sqrt{x}\ =\pm\sqrt{-2}\ \text{ and }\ \pm\sqrt{-4}$$

$$=+j\sqrt{2},-j\sqrt{2},+j2\ \text{ and }\ -j2$$

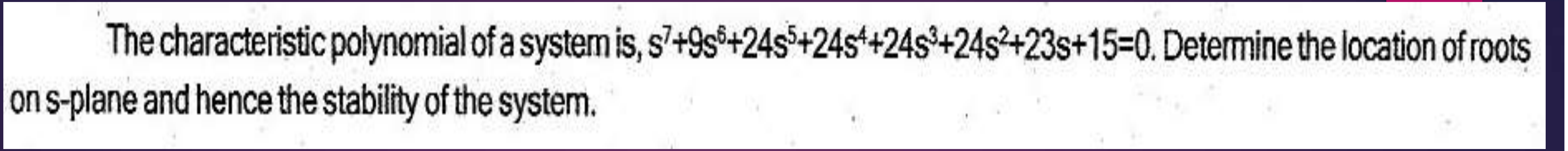


The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

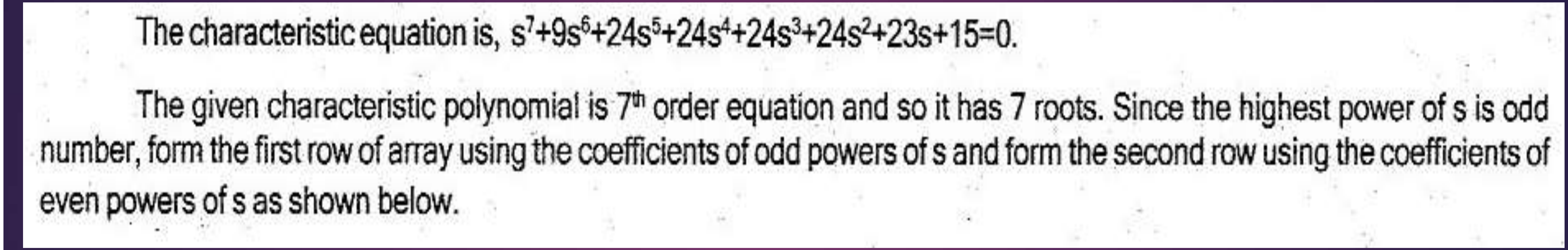
RESULT

1. The system is limitedly or marginally stable.
2. Four roots are lying on imaginary axis and remaining two roots are lying on left half of s-plane.

EX NO:#3

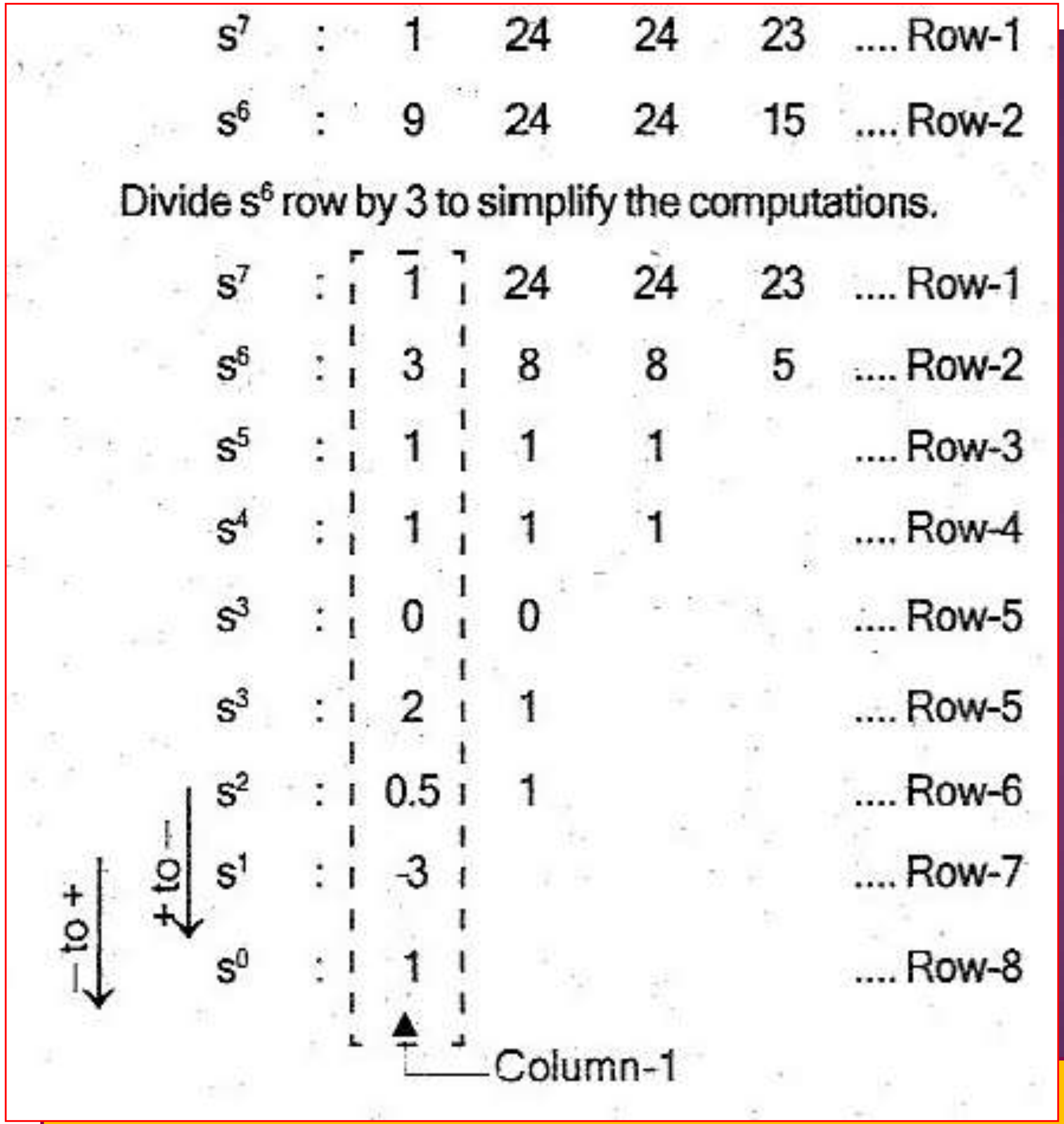


The characteristic polynomial of a system is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.



The characteristic equation is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.



s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

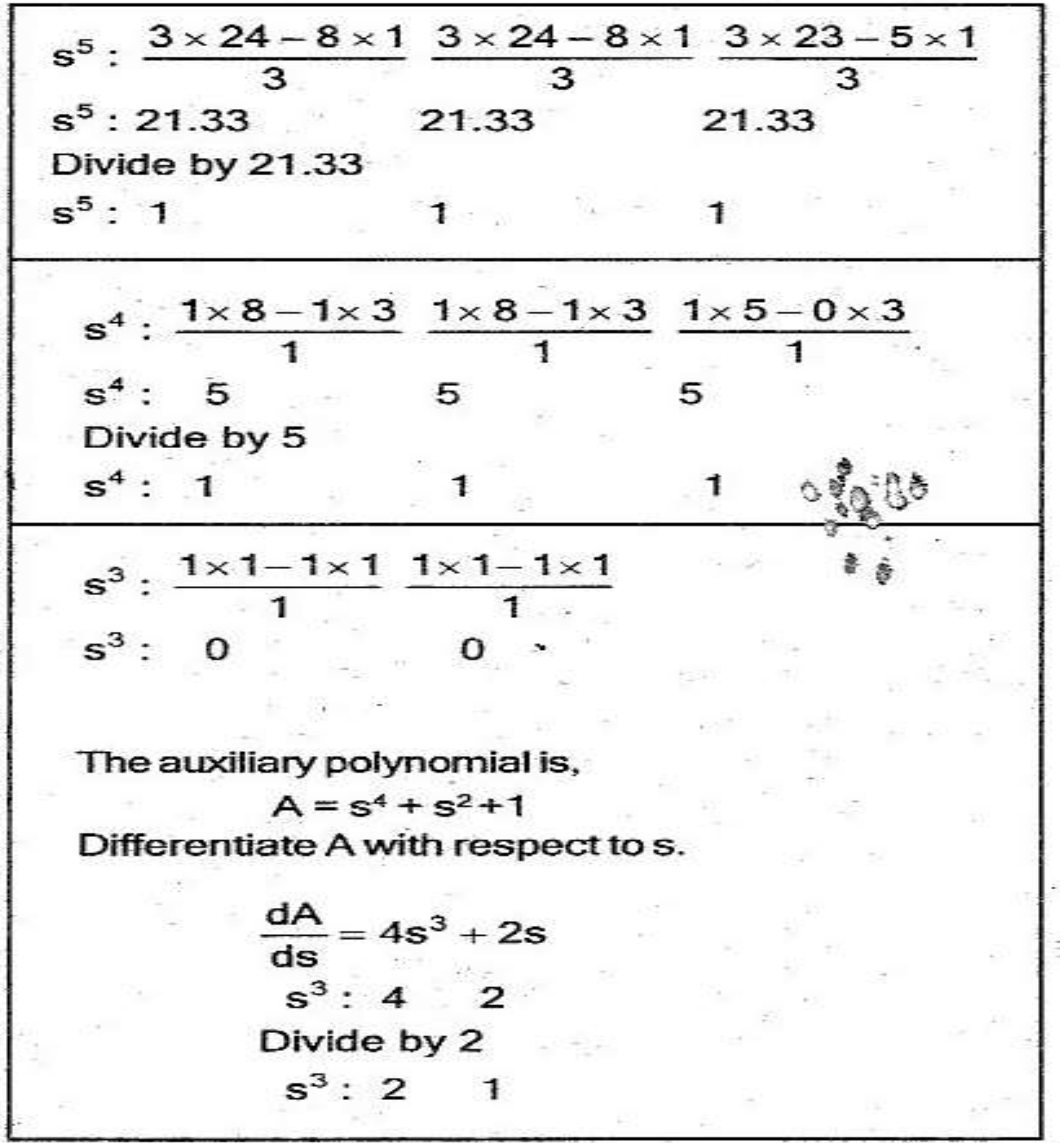
Divide s^6 row by 3 to simplify the computations.

s^7	:	1	24	24	23	.... Row-1
s^6	:	3	8	8	5	.... Row-2
s^5	:	1	1	1		.... Row-3
s^4	:	1	1	1		.... Row-4
s^3	:	0	0			.... Row-5
s^3	:	2	1			.... Row-5
s^2	:	0.5	1			.... Row-6
s^1	:	-3				.... Row-7
s^0	:	1				.... Row-8

+ to –

– to +

Column-1



$s^5 : \frac{3\times24-8\times1}{3} \quad \frac{3\times24-8\times1}{3} \quad \frac{3\times23-5\times1}{3}$

$s^5 :$ 21.33 21.33 21.33

Divide by 21.33

$s^5 :$ 1 1 1

$s^4 : \frac{1\times8-1\times3}{1} \quad \frac{1\times8-1\times3}{1} \quad \frac{1\times5-0\times3}{1}$

$s^4 :$ 5 5 5

Divide by 5

$s^4 :$ 1 1 1

$s^3 : \frac{1\times1-1\times1}{1} \quad \frac{1\times1-1\times1}{1}$

$s^3 :$ 0 0

The auxiliary polynomial is,

$$A = s^4 + s^2 + 1$$

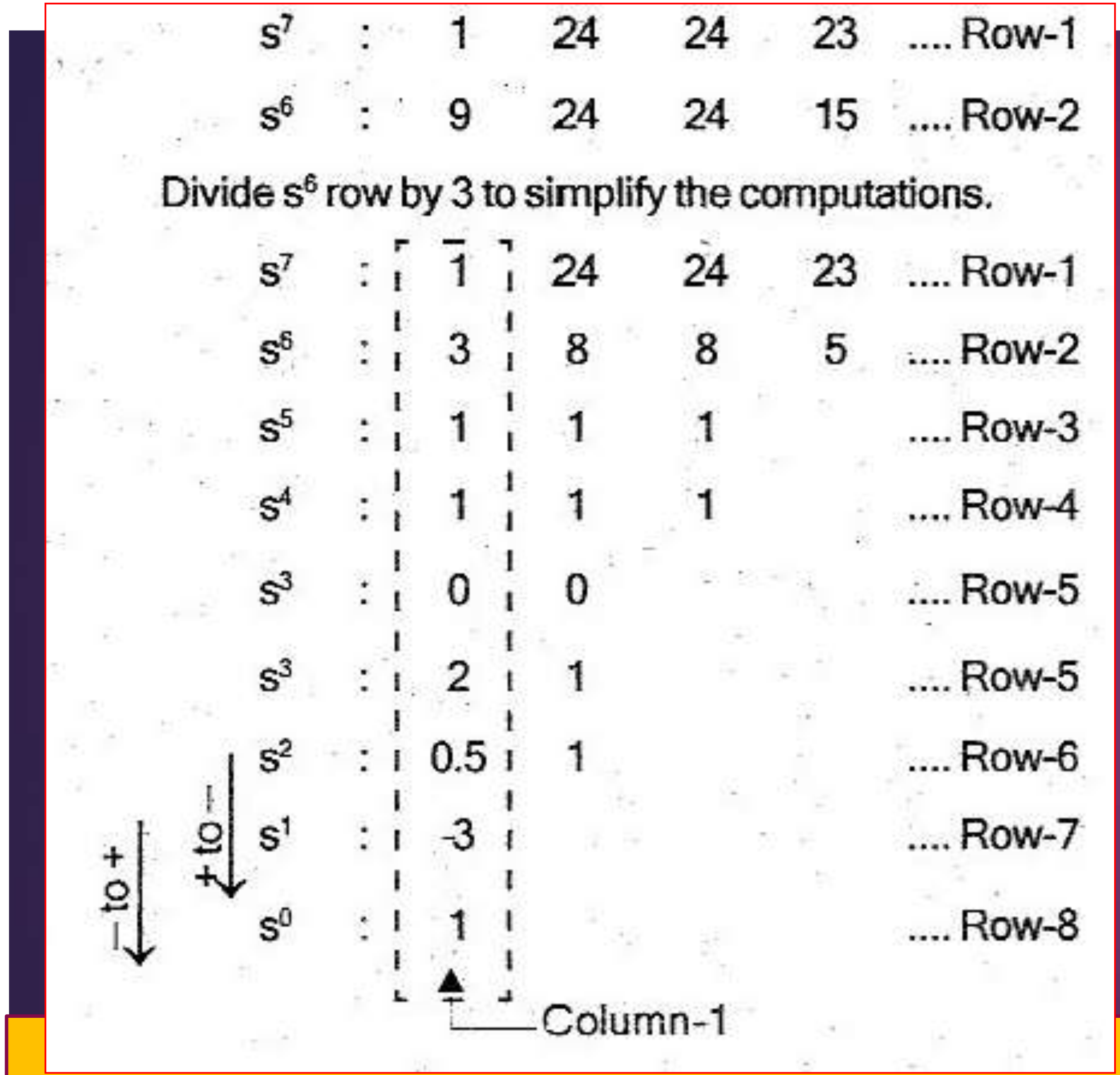
Differentiate A with respect to s.

$$\frac{dA}{ds} = 4s^3 + 2s$$

$s^3 :$ 4 2

Divide by 2

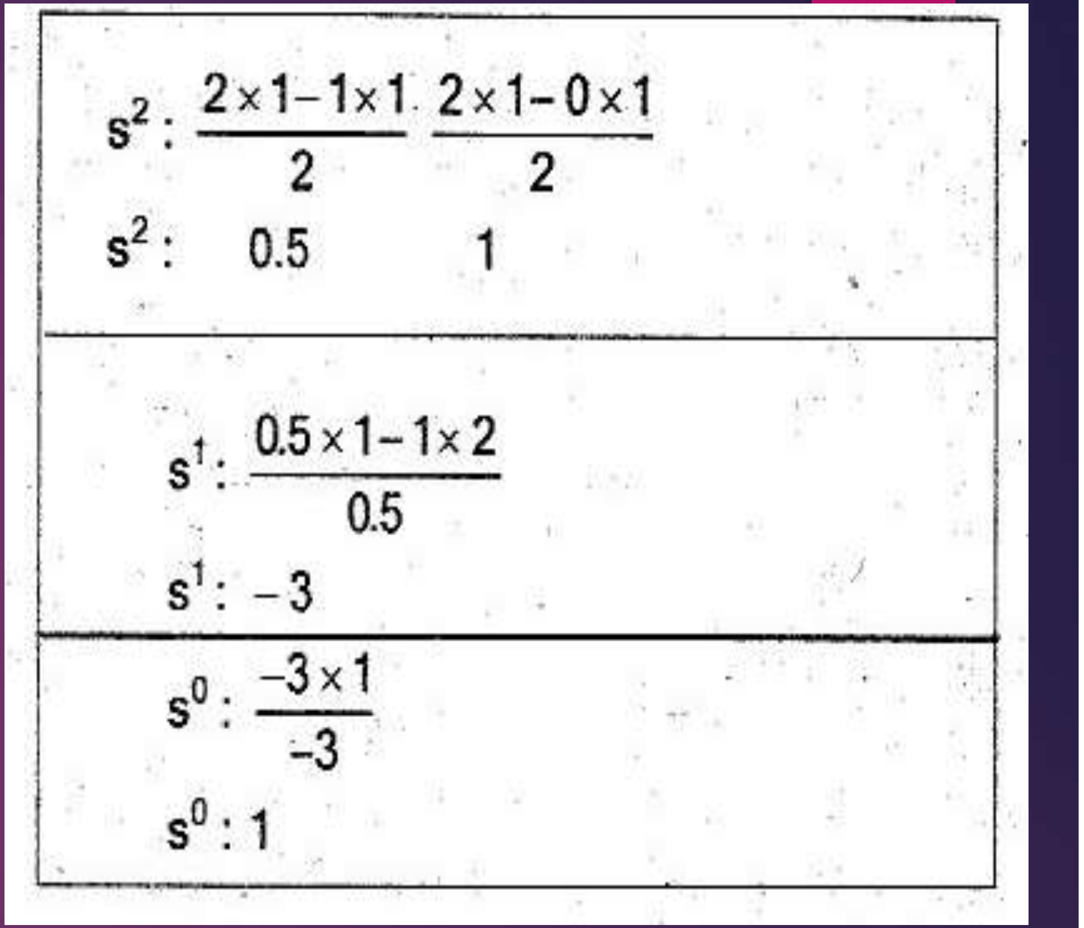
$s^3 :$ 2 1



s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations.

s^7	:	1	24	24	23	.... Row-1
s^6	:	3	8	8	5	.... Row-2
s^5	:	1	1	1		.... Row-3
s^4	:	1	1	1		.... Row-4
s^3	:	0	0			.... Row-5
s^3	:	2	1			.... Row-5
s^2	:	0.5	1			.... Row-6
s^1	:	-3				.... Row-7
s^0	:	1				.... Row-8



$$s^2 : \frac{2\times1-1\times1}{2} \quad \frac{2\times1-0\times1}{2}$$

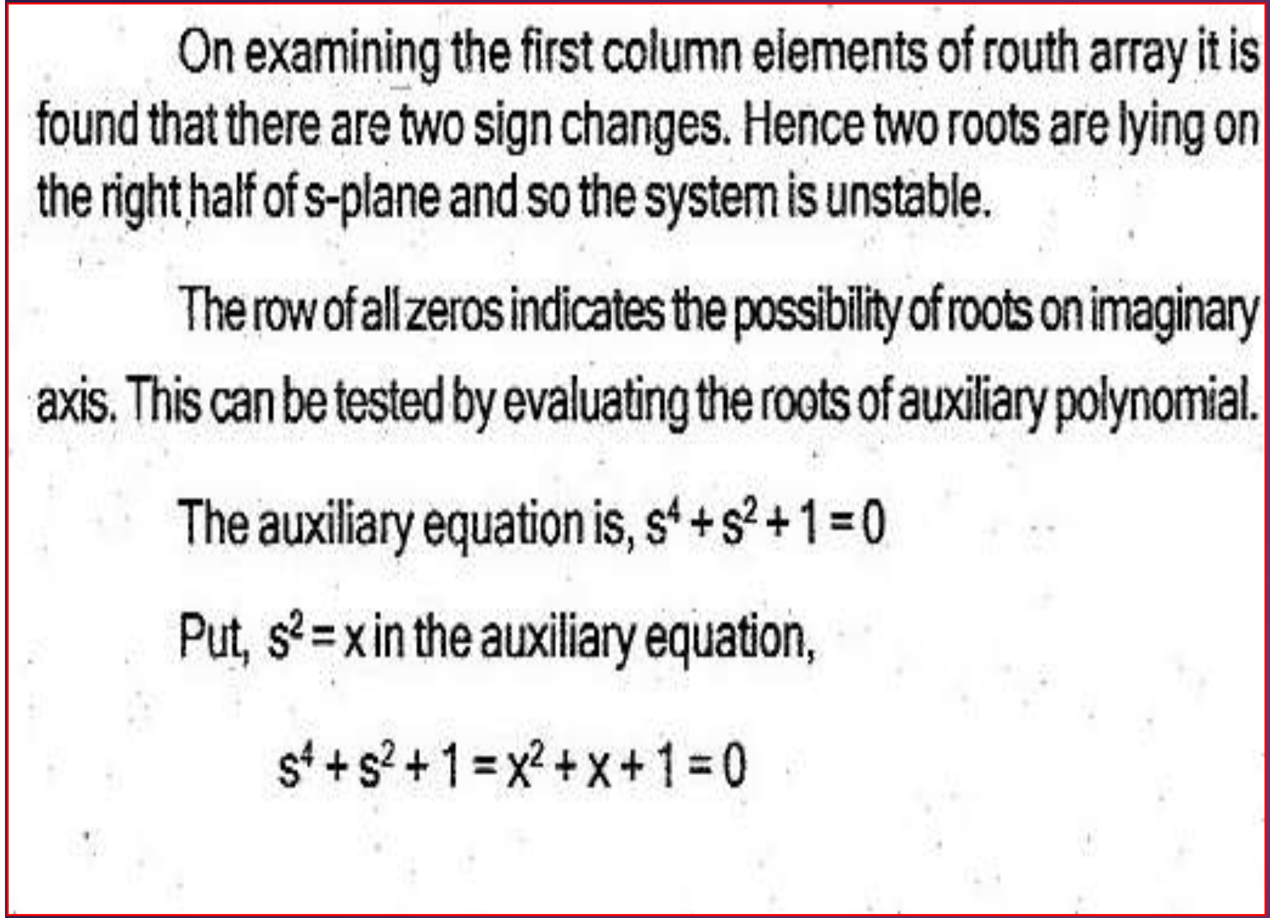
$$s^2 : \quad 0.5 \quad\quad 1$$

$$s^1 : \frac{0.5\times1-1\times2}{0.5}$$

$$s^1 : -3$$

$$s^0 : \frac{-3\times1}{-3}$$

$$s^0 : 1$$



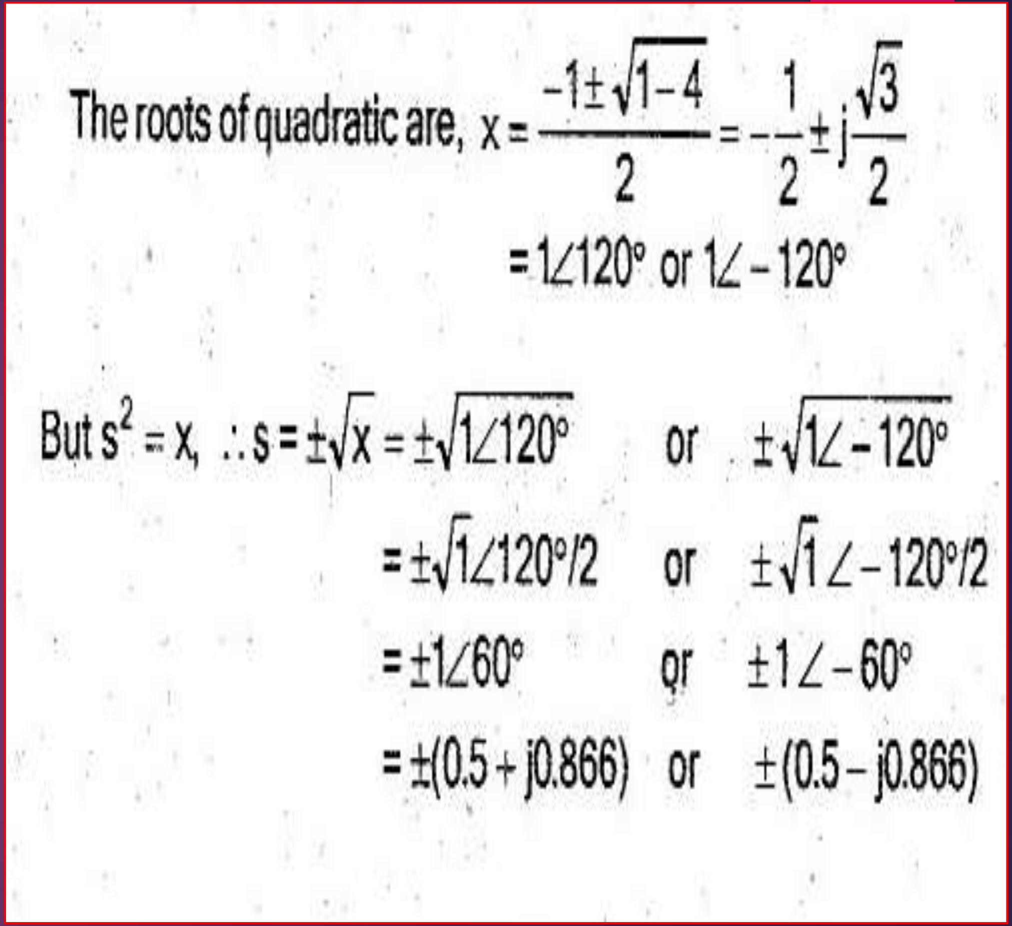
On examining the first column elements of routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and so the system is unstable.

The row of all zeros indicates the possibility of roots on imaginary axis. This can be tested by evaluating the roots of auxiliary polynomial.

The auxiliary equation is, $s^4 + s^2 + 1 = 0$

Put, $s^2 = x$ in the auxiliary equation,

$$s^4 + s^2 + 1 = x^2 + x + 1 = 0$$



The roots of quadratic are, $x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

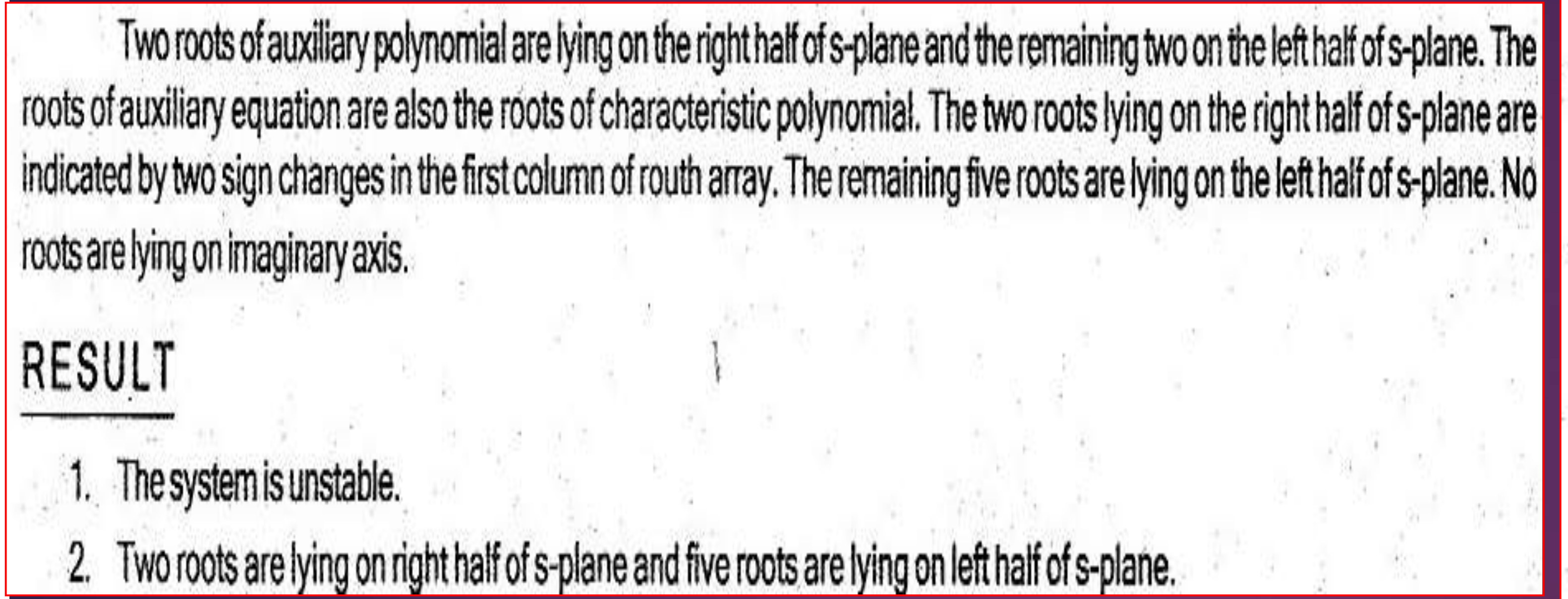
$$= 1\angle 120^\circ \text{ or } 1\angle -120^\circ$$

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ}$ or $\pm\sqrt{1\angle -120^\circ}$

$$= \pm\sqrt{1}\angle 120^\circ/2 \quad \text{or} \quad \pm\sqrt{1}\angle -120^\circ/2$$

$$= \pm 1\angle 60^\circ \quad \text{or} \quad \pm 1\angle -60^\circ$$

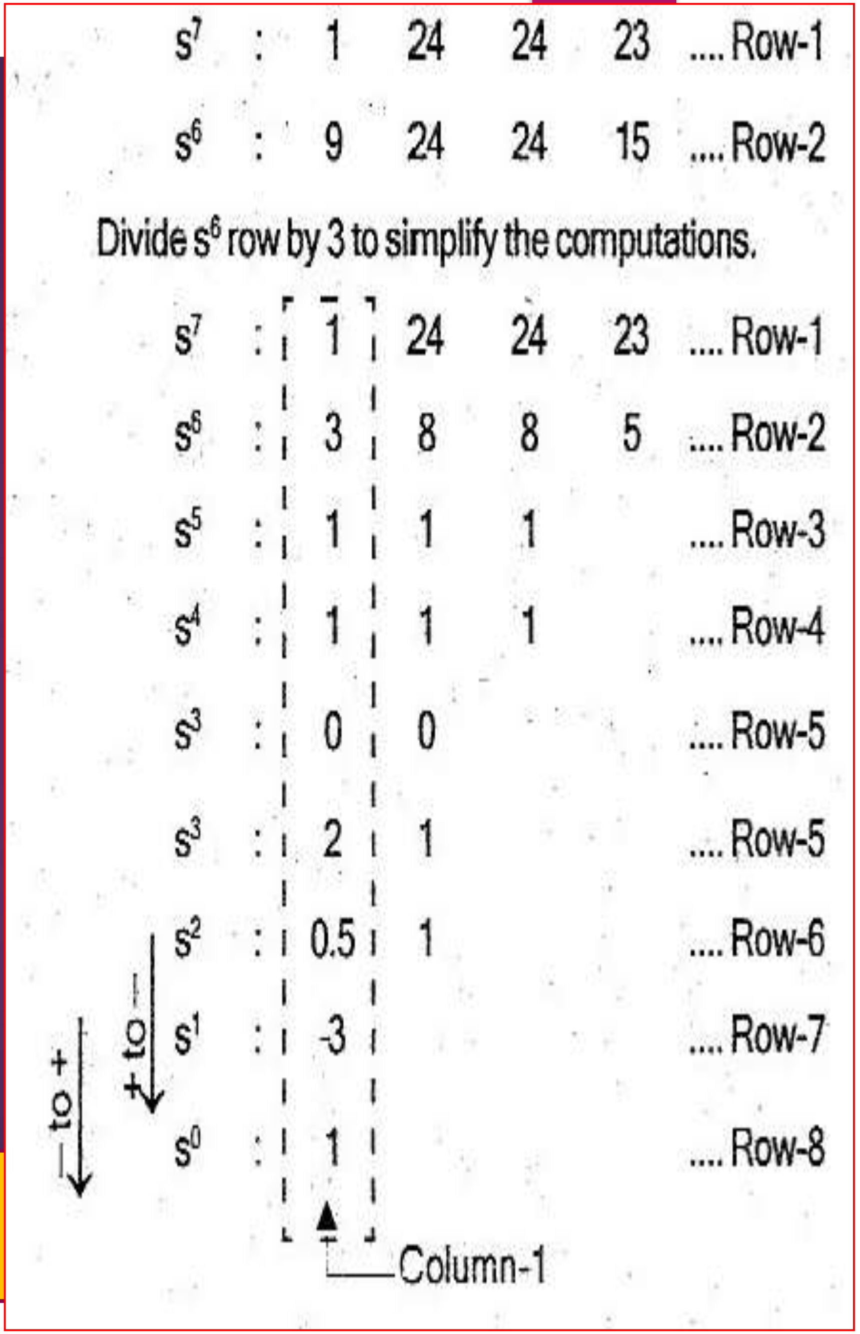
$$= \pm(0.5 + j0.866) \quad \text{or} \quad \pm(0.5 - j0.866)$$



Two roots of auxiliary polynomial are lying on the right half of s-plane and the remaining two on the left half of s-plane. The roots of auxiliary equation are also the roots of characteristic polynomial. The two roots lying on the right half of s-plane are indicated by two sign changes in the first column of routh array. The remaining five roots are lying on the left half of s-plane. No roots are lying on imaginary axis.

RESULT

1. The system is unstable.
2. Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.



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s^3	:	2	1			.... Row-5
s^2	:	0.5	1			.... Row-6
s^1	:	-3				.... Row-7
s^0	:	1				.... Row-8